SEMIPRIME (-1, 1) RINGS

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Abstract
In this paper, we show that in a (-1,1) ring R, every associator commutes with every element of R, that is ((R,R,R),R)=0 and (R,R (R,R,R))=0. Using these we prove that a 2- and 3- divisible semiprime (-1, 1) ring R is associative.

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1. INTRODUCTION

Thedy [1] studied nonassociative rings satisfying the identity ((a, b, c), d) = 0. He proved that a simple nonassociative ring with ((a, b, c), d) = 0 is either associative or commutative. He pointed out that it cannot be extended to prime rings.

In this paper, we show that in a (-1,1) ring R, every associator commutes with every element of R, that is ((R,R,R),R)=0 and (R,R (R,R,R))=0. Using these we prove that a 2- and 3- divisible semiprime (-1, 1) ring R is associative. At the end of this section we give an example of a (-1, 1) ring which is not associative.

2. PRELIMINARIES

A nonassociative ring is said to be a (-1, 1) ring if it satisfies the following identities:

\[ A(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0 \] (1)

and

\[ B(x, y, z) = (x, y, z) + (x, z, y) = 0 \] (2)

We know that a ring R is semi prime if for any ideal A of R, A² = 0 implies A = 0.

A ring R is said to be n – divisible if nx=0 implies x=0 for all x in R and n a natural number.

Throughout this section R denotes a 2- and 3- divisible (-1, 1) ring.

As a consequence of (2), we have the right alternative law (y, x, x) = 0. (3)

In any ring we have the following identities:

\[ c(w, x, y, z) = (wx, y, z) – (w, xy, z) + (w, x, yz) - w(x, y, z) – (w, x, y)z = 0. \] (4)

and

\[ (xy, z) – x(y, z) – (x, z)y – (x, y, z) + (x, z, y) - (z, x, y) = 0. \] (5)

By forming C(x, y, y, z) – C(x, z, y, y) + C(x, y, z, y) = 0,

we obtain 2(x, y, yz) = 2(x, y, z)y. This implies that

\[ D(x, y, z) = (x, y, yz) – (x, y, z)y = 0. \] (6)

In C(x, z, y, y) = 0 we make use of (6),

So that E(x, y, z) = (x, y², z) – (x, y, yz + zy) = 0. (7)
By linearizing (6) (replace y with w + y), we obtain the identity
\[ F(x, w, y, z) = (x, w, yz) + (x, y, wz) - (x, w, z)y - (x, y, z)w = 0. \] (8)

From \( C(w, x, y, z) - F(w, z, x, y) = 0 \), it follows that
\[ G(w, x, y, z) = (wx, y, z) + (w, x, (y, z)) - w(x, y, z) - (w, y, z)x = 0. \]

In a \((-1, 1)\) ring (5) becomes
\[ H(x, y, z) = (xy, z) - x(y, z) - (x, z)y - 2(x, y, z) - (z, x, y) = 0, \]

Because of (2). The combination of (1) and (4) gives
\[ J(w, x, y, z) = (w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y)) = 0. \]

From \( J(x, x, y, z) = 0 \), it follows that
\[ 2(x, (x, y, z)) = 0. \]

From this and the fact that \((x, y, x) = -(x, x, y)\) we obtain
\[ (x, (x, y, x)) = 0 \text{ and } (x, (x, y, x)) = 0. \] (9)

Now \( J(y, x, y, x) = 0 \) gives
\[ 2(y, (x, y, x)) - 2(x, (y, x, y)) = 0. \]

Thus \((y, (x, y, x)) = -(x, (y, x, y))\).

From \( B(x, x, y) = 0 \) and \( B(y, y, x) = 0 \), we have \((y, (x, x, y)) = -(x, (y, y, x)) = 0.\)

Combining this with \( J(y, x, y, x) = 0 \) gives
\[ 2(y, (x, x, y)) = 0 \text{ and therefore } \]
\[ (y, (x, x, y)) = 0. \] (10)

Using the right alternative property of \( R \), identity (10) can be written
\[ (y, (x, x, x)) = 0. \] (11)

Now we define \( U \) to be the set of all elements \( u \) of \( R \) which commute with all the elements of \( R \).

That is, \( U = \{ u \in R : (u, R) = 0 \} \).

Then \( C(x, x, u) = 0 \) gives \( -2(x, x, u) = 0 \).

Hence \( (x, x, u) = 0 \) and \( (x, u, x) = 0 \) by (2).

Replacing \( x \) by \( x + y \) in these last two identities give
\[ (x, y, u) = -(y, x, u) \] (12)
and \( (x, u, y) = -(y, u, x) \), for \( u \in U \). (13)

In addition to these identities, we present some more identities involving the element \( u \in U \).
\[ O = Q(u, x, y) = (u, x, y) - 2(y, x, u) \] (14)
and \[ O = R(x, y, u) = 3(x, y, u) - (x, y)u + (x, y). \] (15)

We know the identity \((y, (x, y, x)) = 0, \) for every \( x, y, \) in \( R \) holds in \( R \). Using this we prove the following lemma.

3. MAIN RESULTS

**Lemma 1:** If \( R \) is a 2- and 3- divisible \((-1, 1)\) ring, then \( ((R, R, R), R) = 0 \).

**Proof:** Using the right alternative property (11) can be written as
\[ (y, (x, x, y)) = 0. \] (16)

By linearizing the identities (11) and (16), we have
\[ (y, (x, y, z)) = -(y, (z, y, x)) \] (17)
and \( (y, (x, z, y)) = -(y, (z, x, y)). \) (18)

From equations (2), (17), (18) and again (2) we get
\[ (y, (y, z, x)) = -(y, (y, x, z)) = (y, (z, x, y)) = -(y, (x, y, z)) = (y, (x, y, z)). \] (19)
Community equation (1) with y, we have
\((y, (x, y, z) + (y, z, x) + (z, x, y)) = 0\). From (19)

This equation becomes \(3(y, (x, y, z)) = 0\). Since \(R\) is 3-divisible,
\((y, (x, y, z)) = 0\). \(\text{(20)}\)

From (20), the identity \(L=(x, (y, z) - 3(y, (x, z, y)) = 0\) in [2] becomes \((x, (y, z)) = 0\).

Thus \((R, (y, y, z)) = 0\). \(\text{(21)}\)

By linearizing equation (21), we obtain \((w, (x, y, z)) = -(w, (y, x, z))\).

Applying equations (2) and (22) repeatedly, we get
\((w, (x, y, z)) = - (w, (y, x, z)) = (w, (y, z, x)) = -(w, (z, y, x)) = (w, (z, x, y)).\)

Commuting equation (1) with \(w\) and applying the above equation, we obtain \(3(w, (x, y, z)) = 0\).

Since \(R\) is 3-divisible, we have \((w, (x, y, z)) = 0\). \(\text{(23)}\)

This completes the proof of the lemma.

**Lemma 2:** If \(R\) is a 2 and 3 divisible (-1, 1) ring, then \((r, w(x, y, z)) = 0\).

**Proof:** Let \(r\) be an arbitrary element of \(R\). By commuting equations (6), (8), (4) with \(r\), and then applying (23) we get
\((r, y(x, z, w)) = -(r, w(x, z, y)), \quad \text{(24)}\)
\((r, y(x, y, z)) = 0 \quad \text{(25)}\)
and
\((r, w(x, y, z)) = -(r, z(w, x, y)). \quad \text{(26)}\)

Linearizing equation (25), we have
\((r, w(x, y, z)) = -(r, z(w, x, y)). \quad \text{(27)}\)

Permutating cyclically \((w z y x)\) in (26) and finally applying (24), we get
\((r, w(x, y, z)) = -(r, z(w, x, y)) = (r, y(z, w, x)) = -(r, x(y, z, w)) = (r, w(y, z, x)). \quad \text{(28)}\)

But using (27) and \(B(x, y, z) = 0\), (28) can be written as
\((r, y(z, w, x)) = -(r, w(z, x, y)) = (r, w(z, x, y)). \quad \text{(29)}\)

Combining (28) and (29) we obtain
\((r, w(x, y, z)) = (r, w(z, x, y)) = (r, w(z, x, y)). \quad \text{(30)}\)

Multiplying equation \(A (x, y, z) = 0\) by \(w\) and commuting with \(r\), and applying (30), then \(3(r, w(x, y, z)) = 0\).

Since \(R\) is 3-divisible, we have \((r, w(x, y, z)) = 0\). \(\text{(31)}\)

Hence this completes the proof of the lemma.

**Theorem 1:** A 2- and 3-divisible semiprime (-1, 1) ring \(R\) is associative.

**Proof:** If \(u\) is an arbitrary associator, from (12) and (2) we have
\((x, y, u) = -(y, x, u) = (y, u, x). \quad \text{(32)}\)

Using (3) and (32) we get
\((u, x, y) = -(u, y, x) = -(y, x, u) = (y, u, x). \quad \text{(33)}\)

From (1) \((x, y, u) + (y, u, x) + (u, x, y) = 0\).

This implies \(3(x, y, u) = 0\) using (32) and (33).

Therefore \((x, y, u) = 0\), since \(R\) is 3-divisible.

Associating equation (4) with \(r, s\) and using \((x, y, u) = 0\), then we obtain
\((r, s, w(x, y, z)) = -(r, s, (w, x, y))z)\)
\(= -(r, s, z(w, x, y)), \quad \text{(34)}\)
\(= (r, s, (z, w, x)y), \quad \text{permuting } z, w, x, y \text{ cyclically}\)
\[(r, s, y(z, w, x)) = (r, s, y(z, x, w)) \text{ using (2).}
\]
\[(r, s, w(y, z, x)) = -(r, s, w(z, x, y)), \text{ using (21).}
\]
\[(r, s, w(z, x, y)) \text{ using (2).}
\]

\[\therefore (r, s, w(x, y, z)) = (r, s, w(y, z, x)) = (r, s, w(z, x, y)) \quad (34)\]

Multiplying the equation (1) with \(w\) and associate with \(r, s\) then we obtain

\[(r, s, w(x, y, z)) + (r, s, w(y, z, x)) + (r, s, w(z, x, y)) = 0.\]

Using (34), the above equation becomes

\[3(r, s, w(x, y, z)) = 0, \text{ since } R \text{ is 3-divisible then we have } (r, s, w(x, y, z)) = 0.\]

We get \((r, s, w) (x, y, z) = 0\) by using (6).

Hence \((R, R, R) (R, R, R) = 0.\)

We know that \(A\) is an associator ideal of \(R\), so \(A.A=0\), since \(R\) is semiprime then the ideal \(A^2 = 0\) implies \(A=0.\)

That is \((R, R, R) = 0.\) Hence \(R\) is associative.

Now we give an example of a \((-1, 1)\) ring, which is nonassociative.

**Example**: Consider the algebra having basis elements \(x, y, z\) over an arbitrary field. We define \(x^2=y, yx=z\) and all other products of basis elements equal to zero. It clearly satisfies (1) and (2) conditions. Hence it is a \((-1, 1)\) ring, but not associative, since \((x, x, x) = z.\)

4. REFERENCES