ON SKOLEM MEAN LABELING FOR FOUR STAR

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ABSTRACT

In this paper, we prove the conjecture that the four star $K_{1,1} \cup K_{1,1} \cup K_{1,m} \cup K_{1,n}$ is a skolem mean graph if $|m-n|<7$ for $m-6 \leq n \leq m$ and $1 \leq m \geq n$.

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1. INTRODUCTION

All graphs in this chapter are finite, simple and undirected. Terms not defined here are used in the sense of Harary [8]. In [2], we proved that the three star $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$ is a skolem mean graph if $|m-n|=4+\ell$ for $\ell=1,2,3,...$, $m=1,2,3,...$, $n=\ell+m+4$ and $\ell\leq m<n$; the three star $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$ is not a skolem mean graph if $|m-n|>4+\ell$ for $\ell=1,2,3,...$, $m=1,2,3,...$, $n\geq \ell+m+5$ and $\ell\leq m<n$; the four star $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$ is a skolem mean graph if $|m-n|=4+2\ell$ for $\ell=2,3,4,...$, $m=2,3,4,...$, $n=2\ell+m+4$ and $\ell\leq m<n$; the four star $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$ is not a skolem mean graph if $|m-n|>4+2\ell$ for $\ell=2,3,4,...$, $m=2,3,4,...$, $n=2\ell+m+5$ and $\ell\leq m<n$; the four star $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$ is a skolem mean graph if $|m-n|=7$ for $m=1,2,3,...$, $n=m+7$ and $1\leq m<n$. Also, the four star $K_{1,\ell} \cup K_{1,m} \cup K_{1,n}$ is a skolem mean graph if $|m-n|>7$ for $m=1,2,3,...$, $n\geq m+8$ and $1\leq m<n$. In [3], the necessary condition for a graph to be skolem mean is that $p\geq q+1$.

2. SKOLEM MEAN LABELING

Definition 2.1: The four star is the disjoint union of $K_{1,a}$, $K_{1,b}$, $K_{1,c}$ and $K_{1,d}$. It is denoted by $K_{1,a} \cup K_{1,b} \cup K_{1,c} \cup K_{1,d}$.

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Definition 2.2 [2]: A graph \( G = (V, E) \) with \( p \) vertices and \( q \) edges is said to be a skolem mean graph if there exists a function \( f \) from the vertex set of \( G \) to \( \{1, 2, 3, \ldots, p\} \) such that the induced map \( f^* \) from the edge set of \( G \) to \( \{2, 3, 4, \ldots, p\} \) defined by

\[
 f^*(e=uv) = \begin{cases} 
 \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even} \\
 \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd, then}
\end{cases}
\]

the resulting edges get distinct labels from the set \( \{2, 3, 4, \ldots, p\} \).

Note 2.3: In a skolem mean graph, \( p \geq q + 1 \).

Theorem 2.4: If \( 1 \leq m \leq n \), the four star \( K_{1,1} \cup K_{1,1} \cup K_{1,m} \cup K_{1,n} \) is a skolem mean graph if \( |m-n| < 7 \) for \( n = 1, 2, 3, \ldots \) and \( m-6 \leq n \leq m \).

Proof:

Case (a) Consider the graph \( G = K_{1,1} \cup K_{1,1} \cup K_{1,m} \cup K_{1,n} \) when \( n = m \). Let us consider the case that \( |m-n| < 7 \) for \( n = 1, 2, 3, \ldots \). We have to prove that \( G \) is a skolem mean graph.

We have \( V(G) = \{u, u_1\} \cup \{v, v_1\} \cup \{w_i : 1 \leq i \leq m\} \cup \{x_j : 1 \leq j \leq n\} \) and
\[
 E(G) = \{uu_1, vv_1\} \cup \{ww_i : 1 \leq i \leq m\} \cup \{xx_j : 1 \leq j \leq n\}.
\]

Then \( G \) has \( m + n + 6 \) vertices and \( m + n + 2 \) edges.

The required vertex labeling \( f : V(G) \to \{1, 2, 3, 4, \ldots, m+n+6\} \) is defined as follows:

\[
 f(u) = 1; \ f(v) = 5; \ f(w) = m+n+5; \ f(x) = 3; \\
 f(u_1) = 9; \\
 f(v_1) = 7; \\
 f(w_i) = 2i \quad \text{for } 1 \leq i \leq m; \\
 f(x_j) = 2j+9 \quad \text{for } 1 \leq j \leq n
\]

The corresponding edge labels are as follows:

The edge label of \( uu_1 \) is 5; \( vv_1 \) is 6; \( ww_i \) is \( \frac{m+n+2i+5}{2} \) for \( 1 \leq i \leq m \) and \( xx_j \) is \( j+6 \) for \( 1 \leq j \leq n \).

Hence the induced edge labels of \( G \) are distinct.

Hence the graph \( G \) is skolem mean graph.

Case-(b): Consider the graph \( G = K_{1,1} \cup K_{1,1} \cup K_{1,m} \cup K_{1,n} \) when \( n = m-1 \). Let us consider the case that \( |m-n| < 7 \) for \( n = 1, 2, 3, \ldots \). We have to prove that \( G \) is a skolem mean graph.

We have \( V(G) = \{u, u_1\} \cup \{v, v_1\} \cup \{w_i : 1 \leq i \leq m\} \cup \{x_j : 1 \leq j \leq n\} \).
\[
 E(G) = \{uu_1, vv_1\} \cup \{ww_i : 1 \leq i \leq m\} \cup \{xx_j : 1 \leq j \leq n\}.
\]

Then \( G \) has \( m + n + 6 \) vertices and \( m + n + 2 \) edges.
The required vertex labeling \( f : V(G) \rightarrow \{1, 2, 3, 4, \ldots, m+n+6\} \) is defined as follows:
\[
\begin{align*}
  &f(u)=1; f(v)=2; f(w)=m+n+5; f(x)=4; \\
  &f(u_1)=6; \\
  &f(v_1)=8; \\
  &f(w_i)=2i+1 \quad \text{for } 1 \leq i \leq m; \\
  &f(x_j)=2j+8 \quad \text{for } 1 \leq j \leq n
\end{align*}
\]

The corresponding edge labels are as follows:

The edge label of \( uu_1 \) is 4; \( vv_1 \) is 5; \( ww_i \) is \( \frac{m+n+2i+6}{2} \) for \( 1 \leq i \leq m \) and \( xx_j \) is \( j+6 \) for \( 1 \leq j \leq n \).

Hence the induced edge labels of G are distinct.

Hence the graph G is skolem mean graph.

**Case-(c):** Consider the graph \( G=K_{1,1} \cup K_{1,1} \cup K_{1,m} \cup K_{1,n} \) when \( n=m-2 \). Let us consider the case that \( |m-n|<7 \) for \( n=1, 2, 3, \ldots \) We have to prove that G is a skolem mean graph.

We have \( V(G)=\{u,u_1\} \cup \{v,v_1\} \cup \{w:1 \leq i \leq m\} \cup \{x:1 \leq j \leq n\} \),
\( E(G)=\{uu_1,vv_1\} \cup \{ww_i:1 \leq i \leq m\} \cup \{xx_j:1 \leq j \leq n\} \).

Then G has \( m+n+6 \) vertices and \( m+n+2 \) edges.

The required vertex labeling \( f : V(G) \rightarrow \{1, 2, 3, 4, \ldots, m+n+6\} \) is defined as follows:
\[
\begin{align*}
  &f(u)=1; f(v)=3; f(w)=m+n+5; f(x)=5; \\
  &f(u_1)=2; \\
  &f(v_1)=7; \\
  &f(w_i)=2i+2 \quad \text{for } 1 \leq i \leq m; \\
  &f(x_j)=2j+7 \quad \text{for } 1 \leq j \leq n
\end{align*}
\]

The corresponding edge labels are as follows:

The edge label of \( uu_1 \) is 2; \( vv_1 \) is 5; \( ww_i \) is \( \frac{m+n+2i+7}{2} \) for \( 1 \leq i \leq m \) and \( xx_j \) is \( j+6 \) for \( 1 \leq j \leq n \).

Hence the induced edge labels of G are distinct.

Hence the graph G is skolem mean graph.

**Case-(d):** Consider the graph \( G=K_{1,1} \cup K_{1,1} \cup K_{1,m} \cup K_{1,n} \) when \( n=m-3 \). Let us consider the case that \( |m-n|<7 \) for \( n=1, 2, 3, \ldots \) We have to prove that G is a skolem mean graph.

We have \( V(G)=\{u,u_1\} \cup \{v,v_1\} \cup \{w:1 \leq i \leq m\} \cup \{x:1 \leq j \leq n\} \),
\( E(G)=\{uu_1,vv_1\} \cup \{ww_i:1 \leq i \leq m\} \cup \{xx_j:1 \leq j \leq n\} \).
Then $G$ has $m+n+6$ vertices and $m+n+2$ edges.

The required vertex labeling $f: V(G) \to \{1, 2, 3, 4, \ldots, m+n+6\}$ is defined as follows:

$\begin{align*}
    f(u) &= 1; f(v) = 2; f(w) = m+n+5; f(x) = 6; \\
    f(u_i) &= 3; \\
    f(v_i) &= 4; \\
    f(w_i) &= 2i+3 \text{ for } 1 \leq i \leq m; \\
    f(x_j) &= 2j+6 \text{ for } 1 \leq j \leq n.
\end{align*}$

The corresponding edge labels are as follows:

The edge label of $uu_i$ is $2$; $vv_i$ is $3$; $ww_i$ is $\frac{m+n+2i+9}{2}$ for $1 \leq i \leq m$ and $xx_j$ is $j+5$ for $1 \leq j \leq n$.

Hence the induced edge labels of $G$ are distinct.

Hence the graph $G$ is skolem mean graph.

Case-(e): Consider the graph $G = K_{i,1} \cup K_{i,1} \cup K_{1,m} \cup K_{1,n}$ when $n = m-4$. Let us consider the case that $|m-n| < 7$ for $n = 1, 2, 3, \ldots$. We have to prove that $G$ is a skolem mean graph.

We have $V(G) = \{u, u_i\} \cup \{v, v_i\} \cup \{w, w_i\;|\;1 \leq i \leq m\} \cup \{x, x_j\;|\;1 \leq j \leq n\}$.

Then $G$ has $m+n+6$ vertices and $m+n+2$ edges.

The required vertex labeling $f: V(G) \to \{1, 2, 3, 4, \ldots, m+n+6\}$ is defined as follows:

$\begin{align*}
    f(u) &= 1; f(v) = 2; f(w) = m+n+5; f(x) = 6; \\
    f(u_i) &= 3; \\
    f(v_i) &= 4; \\
    f(w_i) &= 2i+4 \text{ for } 1 \leq i \leq m; \\
    f(x_j) &= 2j+5 \text{ for } 1 \leq j \leq n.
\end{align*}$

The corresponding edge labels are as follows:

The edge label of $uu_i$ is $2$; $vv_i$ is $3$; $ww_i$ is $\frac{m+n+2i+10}{2}$ for $1 \leq i \leq m$ and $xx_j$ is $j+5$ for $1 \leq j \leq n$.

Hence the induced edge labels of $G$ are distinct.

Hence the graph $G$ is skolem mean graph.

Case-(f): Consider the graph $G = K_{i,1} \cup K_{i,1} \cup K_{1,m} \cup K_{1,n}$ when $n = m-5$. Let us consider the case that $|m-n| < 7$ for $n = 1, 2, 3, \ldots$. We have to prove that $G$ is a skolem mean graph.
We have \( V(G) = \{ u, u_1 \}, \{ v, v_1 \}, \{ w \} \cup \{ w_i : 1 \leq i \leq m \} \cup \{ x_j : 1 \leq j \leq n \} \).

\( E(G) = \{ uu_1, vv_1 \} \cup \{ w_i : 1 \leq i \leq m \} \cup \{ xx_j : 1 \leq j \leq n \} \).

Then G has \( m + n + 6 \) vertices and \( m + n + 2 \) edges. The required vertex labeling 
\( f : V(G) \to \{1, 2, 3, 4, ..., m + n + 6\} \) is defined as follows:

\[
\begin{align*}
    f(u) &= 1; \\
    f(v) &= 2; \\
    f(w) &= m + n + 5; \\
    f(x) &= 5; \\
    f(u_1) &= 3; \\
    f(v_1) &= 4; \\
    f(w_i) &= 2i + 5 \quad \text{for } 1 \leq i \leq m; \\
    f(x_j) &= 2j + 4 \quad \text{for } 1 \leq j \leq n
\end{align*}
\]

The corresponding edge labels are as follows:

The edge label of \( uu_1 \) is 2; \( vv_1 \) is 3; \( w_i \) is \( \frac{m + n + 2i + 10}{2} \) for \( 1 \leq i \leq m \) and \( xx_j \) is \( j + 5 \) for \( 1 \leq j \leq n \).

Hence the induced edge labels of G are distinct.

Hence the graph G is skolem mean graph.

Case-(g): Consider the graph \( G = K_{1,1} \cup K_{1,1} \cup K_{1,m} \cup K_{1,n} \) when \( n = m - 6 \). Let us consider the case that \( |m - n| < 7 \) for \( n = 1, 2, 3, ... \). We have to prove that G is a skolem mean graph.

We have \( V(G) = \{ u, u_1 \}, \{ v, v_1 \}, \{ w \} \cup \{ w_i : 1 \leq i \leq m \} \cup \{ x_j : 1 \leq j \leq n \} \).

\( E(G) = \{ uu_1, vv_1 \} \cup \{ w_i : 1 \leq i \leq m \} \cup \{ xx_j : 1 \leq j \leq n \} \).

Then G has \( m + n + 6 \) vertices and \( m + n + 2 \) edges.

The required vertex labeling 
\( f : V(G) \to \{1, 2, 3, 4, ..., m + n + 6\} \) is defined as follows:

\[
\begin{align*}
    f(u) &= 1; \\
    f(v) &= 2; \\
    f(w) &= m + n + 5; \\
    f(x) &= 6; \\
    f(u_1) &= 3; \\
    f(v_1) &= 4; \\
    f(w_i) &= 2i + 6 \quad \text{for } 1 \leq i \leq m; \\
    f(x_j) &= 2j + 3 \quad \text{for } 1 \leq j \leq n
\end{align*}
\]

The corresponding edge labels are as follows:

The edge label of \( uu_1 \) is 2; \( vv_1 \) is 3; \( w_i \) is \( \frac{m + n + 2i + 11}{2} \) for \( 1 \leq i \leq m \) and \( xx_j \) is \( j + 5 \) for \( 1 \leq j \leq n \).

Hence the induced edge labels of G are distinct.

Hence the graph G is skolem mean graph.
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