A NOTE ON SYSTEMS OF SUMMATION INEQUALITIES

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ABSTRACT

In this paper we discuss some systems of summation inequalities. We also discuss the under and over functions of systems of summation equations.

Keywords: Difference Equation, Summation Equation, Summation Inequality, Under and Over Function.

1. INTRODUCTION:

Agarwal [1], Kelley and Peterson [9] developed the theory of difference equations and difference inequalities. Some difference inequalities and comparison results are obtained by K. L. Bondar [2, 3]. Some summation and difference inequalities are obtained in K. L. Bondar [4, 5]. K. L. Bondar, V. C. Borkar, S. T. Patil [6, 7] and Dang H., Oppenheimer S.F.[8] obtained the existence and uniqueness results for difference equations. Some differential and integral inequalities are given in [10]. In this paper we discuss about systems of summation inequalities. We also discuss the under and over functions of systems of summation equations.

2. PRELIMINARY NOTES

Let \( J = \{ t_0, t_0 + 1 \ldots t_0 + a \} \), \( t_0 \geq 0, t_0 \in \mathbb{R} \), and \( E \) be an open subset of \( \mathbb{R}^n \), consider the difference equations with an initial condition,

\[
\Delta u(t) = g(t, u(t)), \quad u(t_0) = u_0
\]

where \( u_0 \in E, u: J \rightarrow E, g: J \times E \rightarrow \mathbb{R}^n \).

The function \( \phi: J \rightarrow \mathbb{R}^n \) is said to be a solution of initial value problem (1), if it satisfies

\[
\Delta \phi(t) = g(t, \phi(t)), \quad \phi(t_0) = u_0.
\]

The initial value problem is equivalent to the problem

\[
u(t) = u_0 + \sum_{s=t_0}^{t-1} g(s, u(s)).
\]

By summation convention \( \sum_{s=t_0}^{t-1} g(s, u(s)) = 0 \) and so \( u(t) \) given above is the solution of (1).

3. MAIN RESULTS:

Theorem: 3.1 Assume that

(i) \( K: J \times J \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( K(t, s, x) \) is nondecreasing in \( x \) for each fixed \((t, s)\) and one of the inequalities

\[
x(t) \leq h(t) + \sum_{s=t_0}^{t-1} K(t, s, x(s)). \quad (2)
\]

\[
y(t) \geq h(t) + \sum_{s=t_0}^{t-1} K(t, s, y(s)) \quad (3)
\]

is strict where \( x, y: J \rightarrow \mathbb{R}^n \).

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x(t_0) < y(t_0). Then
\[ x(t) < y(t), \quad t \geq t_0. \]  \hfill (4)

**Proof:** Assume that the conclusion (4) is false. Then the set

\[ Z = \bigcup_{i=1}^{n} \{ t \in [t_0, \infty) : x_i(t) \geq y_i(t) \} \]

is nonempty. Let \( t_1 = \inf Z \). By (ii), it is clear that \( t_1 > t_0 \). Furthermore, since \( Z \) is closed, \( t_1 \in Z \), and consequently there exists an index \( j \) such that

\[
\begin{align*}
x_j(t_1) &= y_j(t_1), \\
x_j(t) &< y_j(t), \quad t_0 \leq t < t_1, \\
x_j(t) &< y_j(t), \quad t_0 \leq t < t_1, \quad i \neq j.
\end{align*}
\]

Since \( K \) is monotone nondecreasing in \( x \), it follows that

\[ K_j(t_1, s, x(s)) \leq K_j(t_1, s, y(s)). \]

Hence, using (2) and (3), we arrive at the inequality

\[
\begin{align*}
x_j(t_1) &\leq h_j(t_1) + \sum_{s=t_0}^{t_1-1} K_j(t_1, s, x(s)) \\
&\leq h_j(t_1) + \sum_{s=t_0}^{t_1-1} K_j(t_1, s, y(s)) \\
&< y_j(t_1).
\end{align*}
\]

This is a contradiction to the fact that \( x_j(t_1) = y_j(t_1) \). Hence \( Z \) is empty and the theorem is proved.

Let us now consider the summation operator defined by

\[ K\phi = \sum_{s=t_0}^{t_1-1} K(t, s, \phi(s)). \]  \hfill (5)

**Definition 3.2** We shall say that the operator \( K \) is monotone nondecreasing if, for any \( \phi_1, \phi_2 : J \to \mathbb{R}^n \), such that, for any \( t_1 > t_0 \),

\[
\phi_1(t) < \phi_2(t), \quad t_0 \leq t < t_1,
\]

implies

\[ K \phi_1(t_1) \leq K \phi_2(t_1). \]

**Theorem 3.3** Let the operator \( K \) defined by (5) be monotone nondecreasing. Suppose further that, for \( t > t_0 \),

\[
x - Kx < y - Ky,
\]

where \( x, y : J \times \mathbb{R}^n \). Then \( x(t_0) < y(t_0) \) implies

\[ x(t) < y(t), \quad t \geq t_0. \]  \hfill (6)

**Proof:** Assume that the conclusion of theorem is false. Then set

\[ Z = \bigcup_{i=1}^{n} \{ t \in [t_0, \infty) : x_i(t) \geq y_i(t) \} \]

is nonempty. Let \( t_1 = \inf Z \). By (ii), it is clear that \( t_1 > t_0 \). Furthermore, since \( Z \) is closed, \( t_1 \in Z \), and consequently there exists an index \( j \) such that

\[
\begin{align*}
x_j(t_1) &= y_j(t_1), \\
x_j(t) &< y_j(t), \quad t_0 \leq t < t_1,
\end{align*}
\]
Since $K$ is monotone nondecreasing in $x$ and using above inequalities, it follows that,

$$K_j x_j(t_0) \leq K_j y_j(t_0).$$  \hfill (7)

As a result, (6) and (7) yield

$$x_j(t_1) = x_j(t_0) - K_j x_j(t_0) + K_j y_j(t_1) \leq y_j(t_1).$$

This contradicts the fact that, at $t = t_1$, $x_j(t_1) = y_j(t_1)$, and hence the proof is complete.

**Definition: 3.4** A function $u : J \rightarrow \mathbb{R}^n$ is said to be an under function of the system of summation equation

$$x = j + Kx$$  \hfill (8)

if it satisfies the inequality

$$u < h + Ku.$$ Similarly $u$ is said to be an over function of (8) if verifies the system of inequality

$$u > h + Ku,$$

whereas if $u$ satisfies equation (8), it is said to be a solution of (8).

**Theorem: 3.5** Let the operator $K$ defined by (5) be monotone nondecreasing. Suppose that $x, y, z : J \rightarrow \mathbb{R}^n$ be an under function, a solution and an over function of (8), respectively on $J$. Then

$$x(t_0) < y(t_0) < z(t_0)$$

implies

$$x(t) < y(t) < z(t), \quad t \geq t_0.$$  

**Proof:** As $x(t)$ is an under function and $y(t)$ is a solution of (8) respectively, we have

$$x(t) < h(t) + \sum_{s=t_0}^{t-1} K(t, s, x(s))$$

and

$$y(t) = h(t) + \sum_{s=t_0}^{t-1} K(t, s, y(s)).$$

Also if $x(t_0) < y(t_0)$, then by Theorem 3.1, we have

$$x(t) < y(t), \quad t \geq t_0.$$  

Similarly using definition of solution, an over function of (8) and by Theorem 3.1 again we obtain

$$y(t) < z(t), \quad t \geq t_0.$$  

Hence

$$x(t) < y(t) < z(t), \quad t \geq t_0.$$  

**REFERENCES:**


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