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FULL RANK DECOMPOSITION OF MATRICES AND ITS APPLICATION OVER SKEW FIELDS

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ABSTRACT

Reference full rank factorization theory over fields, this paper will extend the theory of full rank decomposition of matrices over fields to skew fields. This paper also gives some applications, such as the proof of classical inequality and the compute of the Drazin inverse and $M-P$ inverse.

Keywords: *Full rank decomposition; Inequality; Drazin inverse; $M-P$ inverse.*

1. INTRODUCTION

A matrix is expressed by the multiplication of a number of simple structural matrices, which is called the decomposition of matrices. It is important for us to reveal the full rank decomposition and using this theory to solve some problem of matrices. This paper expands the theory of full rank decomposition of matrices. Using this theory to certify Forbeniua inequality and Sylvester inequality and calculate the $Drazin$ inverse and $M-P$ inverse.

In this paper, K denotes a skew field, $K^{m\times n}$ represents the set of $m\times n$ order matrix, $M_n(K)$ represents the set of $n\times n$ order matrix over K, $GL_n(K)$ means the all of the inverse matrix of n order, A^* means the conjugate transpose matrix of A.

2. PREPARATION KNOWLEDGE

Definition: 1[1] Let R is a non-commutative principal ideal domian with involutorial anti-automorphism σ, $A\in R^{m\times n}$, if there exits $X\in R^{n\times m}$ such that:

$$AXA=A,\ XAX=X,\ (AX)^*=AX,\ (XA)^*=XA.$$

Then, we say X is a $Moore-Penrose$ inverse of A, denoted A^+.

Definition: 2[1] Let $A\in M_n(K)$, if there is a $X\in M_n(K)$ such that:

$$A^k=A^{k+1}X,\ X=X^2A,\ AX=XA$$

Then, we say X is a $Drazin$ inverse of A, denoted A^D.

Lemma: 1[2] Let D is a non-singular matrix, $\begin{bmatrix} A & B \\ C & D \end{bmatrix}\in M_n(K)$, then

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix}=r(D)+r(A-BD^{-1}C).$$

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Lemma: 2[1] Let $A \in K^{m\times n}$, $B \in K^{t\times m}$, $C \in K^{s\times n}$ and $rankA = n$, $rankB = m$, $rankC = s$, then,

$$rankA = rankBA = rankAC^T = rankBAC^T$$

Lemma: 3[1] Let $A \in K^{m\times n}$, if there is $T \in M_n(K)$, $S \in M_n(K)$, then

$$rankA = rank(TA) = rank(AS) = rank(TAS)$$

3. MAIN CONCLUSION

Theorem: 1 If $A \in K^{m\times n}$, $rankA = r(r > 0)$, then $A = CD$, where $C \in K^{m\times r}$ is a row full rank matrix, $D \in K^{r\times n}$ is a column full rank matrix.

Proof: Because $rankA = r$, by the series of elementary row transformations, we can transfer A into a row step-formed matrix B, where $B = \begin{pmatrix} D \\ 0 \end{pmatrix}$, $D \in K^{r\times n}$, and $rankA = r$.. Then exists a product of many elementary matrix, such that $PA = B$, namely $A = P^{-1}B$, we write $P^{-1} = \begin{pmatrix} C & M \end{pmatrix}$, $C \in K^{m\times r}$, $M \in K^{m\times(m-r)}$, then A=$\begin{pmatrix} C & M \end{pmatrix}\begin{pmatrix} D \\ 0 \end{pmatrix} = CD$, C is a $m \times r$ column full rank matrix, D is a $r \times n$ row full rank matrix. So $A = CD$ is a column full rank decomposition of A.

Theorem: 2 Let $A \in K^{m\times n}$, $rankA = r(r > 0)$, if A is a line-simplify matrix, the column $i_1, i_2 \cdots i_r$ of matrix D in $B = \begin{pmatrix} D \\ 0 \end{pmatrix}$ is the first r columns of the identify matrix I_r in the proper order. We set $C = \left(\alpha_{i_1}, \alpha_{i_2} \cdots \alpha_{i_r}\right)$, then $A = CD$ is the full rank decomposition of A.

Proof: Actually, finding the full rank decomposition of A is to find the maximal linearly independent group of column vector group $\alpha_1, \alpha_2 \cdots \alpha_n$. Let $C = \left(\alpha_{i_1}, \alpha_{i_2} \cdots \alpha_{i_r}\right)$, then, it is a $m \times r$ column full rank matrix. As to any column vector α_{ij}, we have $\alpha_{ij} = \left(\alpha_{i_1}, \alpha_{i_2} \cdots \alpha_{i_r}\right)\begin{pmatrix} k_{1j} \\ k_{2j} \\ \vdots \\ k_{rj} \end{pmatrix}$,

Note $K = [k_{ij}]_{r\times(n-r)}$, and take n order permutation matrix $\left(\varepsilon_{i_1}, \varepsilon_{i_2} \cdots \varepsilon_{i_n}\right)$, then
$A = (\alpha_1, \alpha_2 \cdots \alpha_n) = \left(\alpha_{i_1}, \alpha_{i_2} \cdots \alpha_{i_r}\right)(I_k, K) = \left(\alpha_{i_1}, \alpha_{i_2} \cdots \alpha_{i_r}\right)[(I_k, K)P_1]$. We let permutation matrix $P_1 = \left(\varepsilon_{i_1}, \varepsilon_{i_2} \cdots \varepsilon_{i_n}\right)^{-1} = \left(\varepsilon_{i_1}, \varepsilon_{i_2} \cdots \varepsilon_{i_n}\right)^H$. $D = (I_r, K)P_1$ is a $r \times n$ row full rank matrix.

For $D = (I_r, K)P_1$ we consider it as follows: according to a series of element row operation, that is there is a universe matrix P, we can transform A into $PA = B$, where $B = (\beta_1, \beta_2 \cdots \beta_n)$,that is $\alpha_i = P^{-1}\beta_i$. We know $(\alpha_1, \alpha_2 \cdots \alpha_n)$ and $(\beta_1, \beta_2 \cdots \beta_n)$ have the same pertinence relativity pertinence. So for $A = P^{-1}B$, we have $B = (\beta_1, \beta_2, \cdots \beta_n) = \left(\beta_{i_1}, \beta_{i_2} \cdots \beta_{i_r}\right)(I_r, K)P$.

So according to a series of element row operations, we can obtain a matrix B as referred in the theory and $B = \begin{pmatrix} I_r \\ 0 \end{pmatrix}(I_r, 0)P_1 = \begin{pmatrix} (I_r, K)P_1 \\ 0 \end{pmatrix} = \begin{pmatrix} D \\ 0 \end{pmatrix}$. $D = (I_r, K)P_1$ is a $r \times n$ row full rank matrix. We take the maximal linearly independent column group $\left(\alpha_{i_1}, \alpha_{i_2} \cdots \alpha_{i_r}\right)$ of A to form a $m \times r$ column full rank matrix $C = \left(\alpha_{i_1}, \alpha_{i_2} \cdots \alpha_{i_r}\right)$.Namely $A = \left(\alpha_{i_1}, \alpha_{i_2} \cdots \alpha_{i_r}\right)[(I_k, K)P_1] = CD$.

Example: 1 Let $A=\begin{bmatrix}1 & i & j & k\\ 0 & i & 0 & k\\ 1 & 0 & j & 0\end{bmatrix}$, compute the column rank decomposition of it. By the series of elementary row transformations, we can transfer A into a row step-formed matrix $B.$ we records the product of elementary matrix.

$$[A,I_3]\to\begin{bmatrix}1 & i & j & k & 1 & 0 & 0\\ 0 & i & 0 & k & 0 & 1 & 0\\ 1 & 0 & j & k & 0 & 0 & 1\end{bmatrix}\to\begin{bmatrix}1 & i & j & k & 1 & 0 & 0\\ 0 & i & 0 & k & 0 & 1 & 0\\ 0 & -i & 0 & -k & -1 & 0 & 1\end{bmatrix}$$

$$\to\begin{bmatrix}1 & i & j & k & 1 & 0 & 0\\ 0 & i & 0 & k & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & -1 & 1 & 1\end{bmatrix}=[B,P]$$

$$B=\begin{bmatrix}1 & i & j & k\\ 0 & i & 0 & k\\ 0 & 0 & 0 & 0\end{bmatrix},\ P=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -1 & 1 & 1\end{bmatrix},\ P^{-1}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1 & -1 & 1\end{bmatrix}$$

According to the theorem 1, we can obtain

$$A=CD=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & i & j & k\\ 0 & i & 0 & k\end{bmatrix}.$$

Example: 2 Let $A=\begin{bmatrix}i & j & k & 0\\ 0 & j & 0 & -i\\ i & 0 & k & j\end{bmatrix}$, compute the column rank decomposition of it. According to the method of theorem 2,

$$A=\begin{bmatrix}i & j & k & 0\\ 0 & j & 0 & -i\\ i & 0 & k & j\end{bmatrix}\to\begin{bmatrix}i & j & k & 0\\ 0 & j & 0 & -i\\ 0 & 0 & 0 & 0\end{bmatrix}\to\begin{bmatrix}1 & -k & j & 0\\ 0 & 1 & 0 & k\\ 0 & 0 & 0 & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & j & -1\\ 0 & 1 & 0 & k\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Then, $A=\begin{bmatrix}i & j\\ 0 & j\\ i & 0\end{bmatrix}\begin{bmatrix}1 & 0 & j & -1\\ 0 & 1 & 0 & k\end{bmatrix}=CD$.

Theorem: 3 Let $A\in M_n(K)$, and $rankA=rank(I_n-A)=n$,then A is a idempotent matrix.

Proof: In order to certify $A^2=A$, only $r(A^2-A)=0$. Let $r(A)=r,$ then $r-n+r(I_n-A)=0$. The full rank decomposition of $A=HL,$ where $H\in K^{m\times r}$ is a row full rank matrix, $L\in K^{r\times n}$ is a column full rank matrix.

Then according to lemma 1:

$$\begin{aligned} r(A^2-A) &= r(HLHL-HL)=r[H(I_r-LI_nH)L]=r(I_r-LI_nH)\\ &=\begin{bmatrix}I_n & H\\ L & I_r\end{bmatrix}-r(I_n)=\begin{bmatrix}I_n-HL & H\\ 0 & I_r\end{bmatrix}-n\\ &=r+r[I_n-HL]-n=r-n+r[I_n-A]=0\end{aligned}$$

So, A is a idempotent matrix.

Theorem: 4 (*Sylvester* inequality) Let $A \in K^{m\times n}$, $B \in K^{n\times t}$, then,

$$rank(AB) \geq rankA + rankB - n$$

Proof: Let $rankA = r(r > 0)$, let $A = HL$, where $H \in K^{m\times r}$ is a row full rank matrix, $L \in K^{r\times n}$ is a column full rank matrix. Then there is $L \in K^{(n-r)\times n}$ such as $\begin{pmatrix} L \\ L_1 \end{pmatrix} \in M_n(K)$, where $rankL_1 = n - r.$

According to lemma 2: $rank(AB) = rank(HLB) = rank(LB) = rank\left[\begin{pmatrix} LB \\ L_1B \end{pmatrix} - \begin{pmatrix} 0 \\ L_1B \end{pmatrix}\right]$. According to lemma 3, we know:

$$rank(AB) \geq rank\left[\begin{pmatrix} L \\ L_1 \end{pmatrix} B\right] - rank(L_1B) = rankB - rank(L_1B)$$

$$\geq rankB - rankL_1 = rankB - (n-r) = rankA + rankB - n .$$

Theorem: 5 (*Forbeniua* inequality) Let $A \in K^{m\times n}$, $B \in K^{n\times t}$, $C \in K^{t\times s}$ then,

$$rank(ABC) \geq rank(AB) + rank(BC) - B$$

Proof: Let $rankB = r(r > 0)$, let $B = HL$, where $H \in K^{m\times r}$ is a row full rank matrix, $L \in K^{r\times n}$ is a column full rank matrix. According to the theorem 4,

$$rank(ABC) = rank(AH)(LC) \geq rank(AH) + rank(LC) - r$$

According to lemma 2, we can know,

$$rank(AH) = rank(AHL) = rank(AB)$$

$$rank(LC) = rank(HLC) = rank(BC)$$

So , $rank(ABC) \geq rank(AB) + rank(BC) - B$.

Theorem:6 Let $A \in M_n(K)$, let $A = HL$, where $H \in K^{m\times r}$ is a row full rank matrix, $L \in K^{r\times n}$ is a column full rank matrix. Then the existences of $A^{\#}$ if and only if LH is a non-singular matrix, and $A^{\#} = H(LH)^{-2}L$.

Proof: If $A^{\#}$ exists, according to $A = A^2X$, we can know $rankA \leq rankA^2$, because $rankA^2 \leq rankA$,

So $$rankA = rankA^2 = r.$$

$$r = rank(HL) = rank(HLHL) = rankH(LH)L \leq rank(HL)$$

$rank(HL) \leq rankL \leq r$, so $rank(LH) = r$. We can know LH is a non-singular matrix.

In addition, because LH is a non-singular matrix, it is meaningful to $X = H(LH)^{-2}L$. It is easy for us to certify X satisfies four conditions of *Drazin* inverse.

The theorem have certified.

Theorem: 7 Let $A \in K^{m\times n}$, let $A = HL$, where $H \in K^{m\times r}$ is a row full rank matrix, $L \in K^{r\times n}$ is a column full rank matrix, then $A^{+} = L^{*}(H^{*}AL^{*})^{-1}H^{*}$.

Proof: First $H^*AL^* = H^*HLL^* = (H^*H)(LL^*)$, and H^*H and LL^* are square matrices. So H^*AL^* is a non-singular matrix.

$$A^+ = L^*(H^*AL^*)^{-1}H^* = L^*(LL^*)^{-1}(H^*H)^{-1}H^*.$$

Let $X = L^*(H^*AL^*)^{-1}H^*$, $AX = H(H^*H)^{-1}H^*$, $XA = L^*(LL^*)^{-1}L$. Then,

$$AXA = A,\ XAX = X,\ (AX)^* = AX,\ (XA)^* = XA.$$

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