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ON A SUBCLASS OF SEMIPOTENT RINGS

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ABSTRACT

A ring R is said to be clean if every element of R is sum of an idempotent and a unit in R. We define a ring R to be a root clean ring if every element of R can be written as a sum of a unit and a square root of 1. In this paper we study root clean rings and its relationship with clean rings and semiboolean rings. Also we obtain some interesting results on semipotent rings.

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***Key Words**: exchange ring, clean ring, strongly clean ring.*

1. INTRODUCTION

Throughout this paper, by a ring R we mean an associative ring with identity. If R is a ring, then $U(R)$, $Id(R)$ and $N(R)$ denote respectively the set of all units, the set of all idempotents and the set of all nilpotent elements of R. We denote the Jacobson radical of R by $J(R)$ and write $T_n(R)$ for the ring of all $n \times n$ upper triangular matrices over R.

An element $a \in R$ is said to be **clean** if $a = e + u$ for some $e \in Id(R)$ and $u \in U(R)$ and is said to be **strongly clean** if in addition e and u commute. An element $a \in R$ is said to be **uniquely strongly clean**(USC, in short) if a can be written uniquely as $a = e + u$ for some $e \in Id(R)$ and $u \in U(R)$ with $eu = ue$. A ring R is said to be **clean**(resp. **strongly clean, uniquely strongly clean**) if every element of R is clean(resp. strongly clean, uniquely strongly clean). A ring R is said to be an **exchange ring** if for every $a \in R$ there exists $e \in Id(R)$ such that $e \in aR$ and $1 - e \in (1 - a)R$. If L is an additive subgroup of a ring R, we say idempotents can be **lifted** modulo L if, for each $x \in R$ with $x - x^2 \in L$ there exists $e \in Id(R)$ such that $e - x \in L$. Call a ring R **semipotent** if every left(equivalently right) ideal not contained in $J(R)$ contains a nonzero idempotent and **potent** if in addition idempotents can be lifted modulo $J(R)$.

It is well known that:

Uniquely strongly clean ⇒ strongly clean ⇒ clean ⇒ exchange ⇒ potent.

A ring R is said to be **semiboolean** if $R/J(R)$ is boolean and idempotents lift modulo $J(R)$. Following M.Alkan et al.[1] an ideal I of a ring R is said to be an **enabling ideal** if $a - e \in I$, for $a \in R$, $e \in Id(R)$ then $a - f \in I$ for some $f^2 = f \in aR$. A ring R is said to be **regular**(resp. **strongly regular**) if for every element $a \in R$ there exists $x \in R$ such that $a = axa$(resp. $a = a^2x$). It is well known that in an abelian ring the notion of regularity and strong regularity coincide.
In this paper we define and study root clean rings and its relationship with clean and semiboolean rings. Also we obtain some interesting results on semipotent rings.

2.ROOT CLEAN RINGS

We recall that an element a in R is said to be square root of 1 if $a^2 = 1$. Camillo and Yu proved the following result[3,Proposition 10]: Let R be a ring in which 2 is invertible, then R is clean if and only if every element of R is a sum of a unit and a square root of 1. This motivated us to study the rings with such a property and we are lead to give the following definition.

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Definition 2.1: An element a in R is said to be **root clean** if a can be written as a sum of a unit and a square root of 1.

Example 2.2: Every nilpotent element in a ring R is root clean, because if $n \in N(R)$ then $1-n=u \in U(R)$ which implies $n=1+(-u)$

Example 2.3: If $e \in Id(R)$ then $2e$ is root clean, because $2e=1+(2e-1)$

Definition 2.4: A ring R is said to be **root clean** if every element of R is root clean.

Example 2.5: Every division ring D with char(D)≠ 2 is root clean, because 1=-1+2 and if 1≠ $a \in D$ then $a=1+(a-1)$

Remark 2.6: No non-trivial boolean ring is root clean. For suppose R is a boolean ring which is root clean. Let $a \in R$ then we have $a=x+u$ for some $x,u \in U(R)$ with $x^2=1$. Note that, since R is boolean $U(R)=1$ and hence 1 is the unique square root of 1. Therefore $a=0$.

Let P be a property which is meaningful for elements of a ring. For any ring R, let $P(R)$={$a \in R$ |a has the property P }. Following Chen[6], a property P is said to be **admissible** if the following hold:

(1) For any ring homomorphism $\sigma: R \to S$, $\sigma(P(R)) \subseteq P(S)$.
(2) For any rings $R \subseteq S$, $P(R) \subseteq P(S)$.
(3) For any $e \in Id(R)$, $P(eRe)$+$P((1-e)R(1-e)) \subseteq P(R)$.

Remark 2.7: In the above definition, condition 2) can be dropped; because 2) follows from 1) by taking σ to be an inclusion map.

If P is an admissible property then following Chen[6], a ring R is said to be ***P*-clean** if every $a \in R$ has the form $a=p+u$ where $p \in P(R)$ and $u \in U(R)$.

Proposition 2.8: Let R be a ring. If $P(R)$={$a \in R | a^2=1$} then P is an admissible property.

Proof: Clearly condition (1) holds. Let $e \in Id(R)$, $a \in P(eRe)$ and $b \in P((1-e)R(1-e))$. Note that $ab=ba=0$. Therefore $(a+b)^2=a^2+b^2$=$e+1-e=1$. Hence $(a+b) \in P(R)$. This completes the proof.

Remark 2.9: A root clean ring is precisely P-clean for the above said admissible property P. Therefore all the results related to P-clean rings obtained in [6] also hold for root clean rings.

The proof of the following result is an easy verification.

Proposition 2.10: If $R=\prod_i R_i$ then R is root clean if and only if each R_i is root clean.

Next we study the relationship between root clean rings and clean, strongly clean and semiboolean rings. First we prove a lemma which improves [3,Proposition 10].

Lemma 2.11: If R is a root clean ring then 2∈ $U(R)$.

Proof: Let R be a root clean ring. So we have $1=x+u$ for some $x,u \in U(R)$ with $x^2=1$
$\Rightarrow x^2-1=0 \Rightarrow (x+1)(x-1)=0 \Rightarrow x=-1$ (because $1-x=u$ is a unit)
Therefore $u=1-x=2$ is a unit in R.

Proposition 2.12: A ring R is clean and 2∈ $U(R)$ if and only if R is root clean.

Proof: Proof follows from [3,Proposition 10] and lemma 2.11 above.

Corollary 2.13: Every root clean ring is clean.

Our next result shows that a root clean ring can never be semiboolean.

Proposition 2.14: R is semiboolean if and only if R is clean and $J(R)$ equals the set of all quasi regular elements.

Proof: Only if: First we prove that $J(R)$ equals the set of all quasi regular elements. Since every element of $J(R)$ is quasi regular, it is sufficient to prove that every quasi regular element is in $J(R)$. Let $a \in R$ be quasi regular element.

Then $a = 1 - u$ for some $u \in U(R)$ which implies $\overline{1-a} = \bar{u}$ is a unit in $R/J(R)$. Since R is semiboolean, $R/J(R)$ is boolean. Therefore $\overline{1-a} = \bar{1}$ in $R/J(R)$ which implies $a \in J(R)$. Now we show that R is clean. Let $a \in R$. Since $R/J(R)$ is boolean, we get that $\bar{a}$ is an idempotent in $R/J(R)$. Since by hypothesis idempotents lift modulo $J(R)$, we have $\bar{a} = \bar{e}$ for some $e \in Id(R)$ which implies $\bar{a} = -\bar{e} = \overline{-e}$ (because R boolean implies $a = -a$). Therefore we get $a + e \in J(R)$ which implies $a + e = 1 + u$ for some $u \in U(R)$ which in turn implies $a = 1 - e + u$. Hence R is clean.

If: We first show that $R/J(R)$ is boolean. Let $a \in R$.

Claim: $\bar{a}$ is an idempotent in $R/J(R)$.

If $\bar{a} = \bar{0}$ then there is nothing to prove. So suppose that $\bar{a} \neq \bar{0}$. Therefore by hypothesis we get $a = 1 - e + u$ for some $0 \neq e \in Id(R)$ and $u \in U(R)$ which implies $\bar{a} = \overline{1-e} + \bar{u}$. Since by hypothesis $J(R)$ equals the set of all quasi regular elements, we get $\bar{u} = \bar{1}$ in $R/J(R)$. Therefore $\bar{a} = \overline{1-e} + \bar{1} = \overline{-e} = \bar{e}$ (because 2 being quasi regular element is in $J(R)$ which implies $2e \in J(R)$). Therefore $\bar{a}$ is an idempotent in $R/J(R)$. Now by hypothesis , R is clean which implies idempotents lift modulo $J(R)$ and hence R is semiboolean. This completes the proof.

Corollary 2.15: A root clean ring is never semiboolean.

Proof: Suppose that R is root clean which is also semiboolean. Then by above proposition $2 \in J(R)$. But by Lemma 2.11, $2 \in U(R)$ which is a contradiction.

Definition 2.16: An element a in a ring R is said to be **Strongly root clean** if $a = x + u$ for some $x, u \in U(R)$ with $x^2 = 1$ and $xu = ux$. A ring R is said to be **strongly root clean** if every element of R is strongly root clean.

Next result gives examples of strongly root clean rings and also establishes the relation between strongly root clean rings and strongly clean rings.

Proposition 2.17[10,Proposition 5]: A ring R is strongly clean and $2 \in U(R)$ if and only if every element is the sum of a unit and a square root of 1 which commute(i.e, R is strongly root clean).

Proposition 2.18: If $e \in Id(R)$ and R is strongly root clean then so is eRe.

Proof: Let $e \in Id(R)$ and R be a strongly root clean ring. Now by above proposition R is strongly clean and $2 \in U(R)$. Therefore by[5, Theorem 2.4] we get that eRe is strongly clean. Since 2 is a central unit in R, we have $2e \in U(eRe)$. Hence by above proposition eRe is strongly root clean.

Proposition 2.19: If R is a strongly root clean ring then every element a in R with the property $a^2 = 1$ can be written uniquely as a sum of a unit and a square root of 1 which commute.

Proof: Let $a \in R$ with $a^2 = 1$. Note that by Lemma 2.11, 2 is a unit in R . Therefore $a = (-a) + 2a$ where $(-a)^2 = 1$ and $2a \in U(R)$ and they commute with each other. Suppose $a = x + u$ for some $x, u \in U(R)$ with $x^2 = 1$ and $xu = ux$ then $1 = xa + ua = (1 + ux) + ua$ (because $a = x + u \Rightarrow xa = x^2 + ux = 1 + ux$) which implies $x = -a$ and hence $u = 2a$. This completes the proof.

Proposition 2.20: If R is a ring with the property: for each $a \in R$, $a = x + u$ for some $x, u \in U(R)$ with $x^2 = 1 = u^2$ and $xu = ux$, then R is commutative regular ring(in the sense of von-Neumann) with char(R)=3.

Proof: First we note that by above proposition, identity element of R can be written uniquely as 1=(-1)+2. Therefore by hypothesis we have, $2^2 = 1$ which implies that char(R)=3. Now we prove that R is commutative. Let $a \in R$. By hypothesis we have $a = x + u$ for some $x, u \in U(R)$ with $x^2 = 1 = u^2$ and $xu = ux$. It is easy to show that $a^m = 2^{m-1}(1 + xu)$ if m is even and $a^m = 2^{m-1}a$ if m is odd. Since $2^2 = 1$, we have $a^3 = a$ for each $a \in R$-------(1). In particular $u^3 = u$ for each $u \in U(R)$ which implies $u^2 = 1$ for each unit u in R. Now let $u, v \in U(R)$. Therefore $uv \in U(R)$ and hence $(uv)^2 = 1$ which implies $uvuv = 1$ which in turn implies that $uv = vu$ that is, in R any two units commute. Since by hypothesis R is root clean we get that R is commutative. Now by (1) it follows that R is regular(in the sense of von-Neumann). This completes the proof.

3. RINGS SATISFYING (*)

Following G.Borooah et al.[2], given $e \in Id(R)$ and $a \in R$, we say that a is ***e*-clean** if $a - e \in U(R)$ and a is **strongly *e*-clean** if in addition a and e commute.

Note that an element a in a ring R is USC if and only if it is strongly e-clean for a unique idempotent e in R.

We say that a ring R satisfies (*) if the following holds: if $a \in R$ is not right invertible then $e^2 = e \in aR \Rightarrow e = ax = xa$ for some $x \in R$--------------(*)

We recall that a ring R is said to have **stable range one** if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by \in U(R)$ and is said to be **directly finite** if for any $a, b \in R$, $ab = 1$ implies $ba = 1$. It is well known that a ring having stable range one is always directly finite.

Lemma 3.1: If a ring R satisfies (*) then R is directly finite.

Proof: Let $a, b \in R$ be such that $ab = 1$. Note that $(ba)^2 = b(ab)a = ba$ which implies $ba \in Id(R)$. If b is right invertible then there is nothing to prove. So suppose that b is not right invertible. Since R satisfies (*) and $ba \in bR$ we get that $ba = bx = xb$ for some $x \in R$. This implies $a = (ab)a = a(ba) = a(bx) = (ab)x = x$. Therefore $ba = ab = 1$ which is a contradiction. This completes the proof.

Remark 3.2: The converse of the above lemma need not be true. Example is given in Remark 3.5.

Proposition 3.3: If R satisfies (*) then the following are equivalent:
(1) R is an exchange ring.
(2) R is a clean ring
(3) R is a strongly clean ring.

Proof: 3)⇒ 2): Obvious
(2)⇒ (1): This follows from [8,Proposition 1.8(1)].

(1)⇒ (3): Let R be an exchange ring and let $a \in R$. Suppose either a or $(1 - a)$ is right invertible. Then by Lemma 3.1 above, either a or $(1 - a)$ is a unit and hence a is either strongly 0-clean or strongly 1-clean. So suppose that neither a nor $(1 - a)$ is right invertible. Since by hypothesis R is exchange, there exists $e \in Id(R)$ such that $e \in aR$ and $(1 - e) \in (1 - a)R$. Now by (*) we get $e = ax = xa$ and $1 - e = (1 - a)y = y(1 - a)$ for some $x, y \in R$.

Therefore by the proof of [5, Theorem 2.2], we see that a is strongly $(1 - e)$-clean. This completes the proof.

For convenience we denote, the set of all quasi regular elements which are USC, by $Q_{usc}(R)$.

Proposition 3.4: If R is a semipotent ring satisfying (*) then $Q_{usc}(R) = J(R)$

Proof: Let $a \in Q_{usc}(R)$. Therefore we have, a is USC with $a = 1 + u$ for some $u \in U(R)$. We prove that $a \in J(R)$. Suppose a is not in $J(R)$. Since R is a semipotent ring there exists $0 \neq e \in Id(R)$ such that $e \in aR$. Since R satisfies (*) above we have $e = ax = xa$ for some $x \in R$. Since $a \in Q_{usc}(R)$ we have $1 - a \in U(R)$ and by the fact that e commutes with a we get $1 - e$ commutes with $(1 - a)$. Therefore we have $e = ax = xa$ and $1 - e = (1 - a)y = y(1 - a)$ for some $x, y \in R$. Therefore by the proof of [5,Theorem 2.2] we see that a is strongly $(1 - e)$-clean. Since a is USC we have $e = 0$ which is a contradiction. Therefore $a \in J(R)$ and hence $Q_{usc}(R) \subseteq J(R)$. Note that $J(R) \subseteq Q_{usc}(R)$ is always true. For, let $a \in J(R)$ and let a be strongly e-clean. Then we have $a(1 - e) = (a - e)(1 - e)$ which implies $(1 - e) = a(1 - e)(a - e)^{-1} \in aR \subseteq J(R)$. Therefore $e = 1$ and hence $J(R) \subseteq Q_{usc}(R)$

Remark 3.5: The converse of the above proposition need not be true.
Example: Here we give examples of rings for which $Q_{usc}(R) = J(R)$ holds but (*) does not hold. Let R be a commutative uniquely clean ring. Then by [4, Theorem 10], $S = T_n(R)$ is USC. Since every USC ring is exchange, idempotents can be lifted modulo $J(S)$ and by [4, Corollary 18], $S/J(S)$ is boolean. Therefore S is semiboolean.

Therefore by proposition 2.14, $J(S)$ equals the set of all quasi regular elements. Since S is USC, we get $Q_{usc}(S) = J(S)$. But S does not satisfy (*). For, let $A, B \in T_n(R)$ be such that $A = E_{11}$ and $B = E_{11} + E_{12}$. Then clearly B is an idempotent in $T_n(R)$ such that $B = AB \in AT_n(R)$ but $B \neq CA$ for any $C \in T_n(R)$.

Note that this example also shows that the converse of lemma 3.1 need not be true. For, if S is as above then as noted earlier $S/J(S)$ is boolean and hence $S/J(S)$ has stable range one. Therefore by [9,Theorem 2.2], S has stable range one and hence S is directly finite. Thus $S = T_n(R)$ in the above example is directly finite ring which does not satisfy (*).

Corollary 3.6: If R is semipotent ring satisfying (*) then $R/J(R)$ is reduced.

Proof: Let R be a semipotent ring satisfying (*) and let $\bar{a}$ be a nilpotent element in $R/J(R)$. Therefore $a^n \in J(R)$ for some natural number n which implies $1 - a^n \in U(R)$ which in turn implies $(1 - a)(1 + a + a^2 \ldots + a^{n-1}) \in U(R)$.

Therefore $(1-a) \in U(R)$, that is, a is quasi regular and is strongly 1-clean. Now we prove that a is USC so that

$a \in Q_{usc}(R)$. For this we need only prove that a is strongly e-clean for a unique idempotent e in R. Suppose a is strongly e-clean. Then we have $e+{}^nC_1eu+\ldots+{}^nC_{n-1}eu^{n-1}+u^n=(e+u)^n=a^n \in J(R)$ where $u=a-e \in U(R)$. Therefore $(1-e)a^n=(1-e)u^n \in J(R)$ which implies $(1-e) \in J(R)$. Therefore $e=1$ and hence a is USC. Thus $a \in Q_{usc}(R)=J(R)$, by the above proposition. This completes the proof.

Proposition 3.7: Let R be a ring such that idempotents can be lifted modulo $J(R)$ then the following holds: if R satisfies (*) then so does $R/J(R)$.

Proof: Let R be a ring satisfying (*) such that idempotents can be lifted modulo $J(R)$. Let $\bar{a} \in R/J(R)$ be not right invertible and $(\bar{a})^2=\bar{a} \in \bar{a}\bar{R}$. Since by hypothesis idempotents can be lifted modulo $J(R)$, we get $\bar{a}=\bar{e}$ for some $e \in Id(R)$. Therefore $ar-e \in J(R)$ for some $r \in R$. Note that in any ring R, $J(R)$ is enabling ideal by[1, Proposition 5].
Therefore there exists $f \in Id(R)$ such that $ar-f \in J(R)$ with $f \in arR \subseteq aR$. Since $\bar{a}$ is not right invertible in $R/J(R)$, we get a is not right invertible in R. Therefore by (*) we have $f=ax=xa$ for some $x \in R$ which implies $\bar{f}=\bar{a}\bar{x}=\bar{x}\bar{a}$. Note that $ar-f \in J(R) \Rightarrow \bar{f}=\bar{a}\bar{r}=\bar{e}=\bar{a}$. Hence $R/J(R)$ satisfies (*).

Note that in the proof of the above proposition, two important properties of $J(R)$ which we have used are lifting and enabling properties. So, the same proof holds good if we replace $J(R)$ by any ideal satisfying these two properties. Since, by [8, Corollary 1.3] and [1, Example 4], every ideal in an exchange ring has these two properties, we have the following result.

Proposition 3.8: R is an exchange ring satisfying (*) if and only if so is every homomorphic image of R.

Proposition 3.9: If R is as an exchange ring satisfying (*) then R has stable range one.

Proof: We first note that, by [9,Theorem 2.2] a ring R has stable range one if and only if $R/J(R)$ has stable range one.

Now R being an exchange ring, it is semipotent and hence by corollary 3.6 $R/J(R)$ is reduced and hence abelian. Since by [11,Theorem 6] every abelian exchange ring has stable range one, we get that R has stable range one.

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