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A GENERALIZATION OF HAHN-BANACH THEOREM

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ABSTRACT

We present a generalization of Hahn-Banach extension theorem. In this paper, we introduce the notion of s-convex function, and provide an proof for the new version of the Hahn-Banach theorem which says that if a linear operator defined on a subspace X_0 of a real vector space X is dominated by a s-convex function defined on X, then it has a linear extension which is also dominated by the same s-convex operator defined on X.

Key words: *Hahn-Banach theorem, linear subspaces, s-convex function.*

AMS subject classifications: *90C25, 90C33*

1. INTRODUCTION:

The Hahn-Banach theorem is one of the three most important and fundamental theorems in basic Functional Analysis, the other two being the Uniform Boundedness Principle and the Closed Graph Theorem. Usually Hahn-Banach theorems are taught before the other two and most books also present Hahn-Banach Theorems ahead of Uniform Boundedness Principle or the Closed Graph Theorem, see [1] and the references therein. This may be due to several reasons. The statements, proofs and applications of Hahn-Banach theorems are relatively easier to understand. In particular, the hypotheses do not include completeness of the underlying normed linear spaces and proofs do not involve the use of Baire's Category Theorem.

There are two classes of theorems commonly known as Hahn-Banach theorems, those are Hahn-Banach theorems in the extension form and Hahn-Banach theorems in the separation form. All these theorems assert the existence of a linear functional with certain properties. Why is it important to know the existence of such functional? In a large amount of applications of practical importance, the objects of study can be viewed as members of a vector space. A study of such objects involves making various measurements/observations. These are functional on that vector space. As the names suggest, Hahn-Banach theorems in the extension form assert that functional defined on a subspace of a vector space(frequently with some additional structure, usually with a norm or a topology) and having some additional properties(like linearity, continuity) can be extended to the whole space while retaining these additional properties. This is useful in asserting the existence of certain functional and this in turn can be used in applications involving approximation of certain functions. These theorems and their proofs are analytic in nature. On the other hand, Hahn-Banach theorems in the separation form and also their proofs are geometric in nature.

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In this paper, we prove Hahn-Banach theorem in the former form, i.e Hahn-Banach theorem in the extension form, in general, it is called Hahn-Banach theorem, and Hahn-Banach theorem in the latter form is called the separation theorem, see [1].

It is well known that the importance of the Hahn-Banach theorem in Functional Analysis. Hahn-Banach theorems are essentially theorems about real vector spaces. Basic theorems are first proved for real vector spaces. These are then extend to the case of complex vector space by virtue of a technical result (See Lemma7.1 of [2] and remarks preceding it.). In several papers, the authors confine themselves to real vector spaces. Common examples of real vector spaces are $\mathbb{R}$, $\mathbb{R}^n$, sequence spaces ℓ^p ($1 \le p \le \infty$), function spaces $C[a,b]$ with point-wise or coordinate-wise operations as the case may be. Many authors have done a lot of important work on Hahn-Banach theorem. Anger and Lembcke [4] studied Hahn-Banach type theorems for hypo-linear functional (i.e. sub-linear functional which may attain the value $+\infty$), defined on a sub-cone of a locally convex topological vector space. Wang [6] gave a separation theorem of convex cone on an ordered vector space. Theirfelder [7] introduced a general separation concept for sets in product spaces in $X \times Z$ where Z is partially ordered, and formulated some concrete separation theorems using some results about non-vertical affine manifolds. Thierfelder [8] obtained several separation and extension theorems in product spaces $X \times Z$ where Z is partially ordered, and in order to give general assertions suitable for using in vector optimization they replaced the order relation "$\le$" by "$\not>$". Meng [5] generalized Hahn-Banach theorem by using the concept of efficient for K-convex multifunction and K-sublinear multifunction in partially ordered locally convex topological vector space. Simons [9] discussed a new version of the Hahn-Banach theorem, with applications to linear and nonlinear functional analysis, convex analysis, and the theory of monotone multifunction. Peng et al. [10] introduced the notion of affine-like set-valued maps, presented some properties of these maps and proved a new Hahn-Banach extension theorem with a K-convex set-valued map dominated by an affine-like set-valued map. Peng et al. [11] generalized the Hahn-Banach theorem from scalar or vector-valued case to set-valued case. Zalinesu [12] obtained several generalizations of the Hahn-Banach extension theorem to K-convex multifunction. Haddadi and Mazaheri [13] introduced a new extension to Hahn-Banach theorem and considered its relation with the linear operators, and gave some applications of this theorem at the end.

The classical Hahn-Banach theorem is in the vector space. A functional on such a vector space is a real valued function defined on it. It was not a surprise that the generalization of this theorem to the case when $\mathbb{R}$ is replaced by an ordered linear space having the least upper bound property, the so called Hahn-Banach-Kantorovich theorem, interested many mathematicians. The scope of this paper to discuss Hahn-Banach type extension theorems involving single-valued when $\mathbb{R}$ is replaced by an ordered linear space having the least upper bound property. In particular, this ordered linear space is induced by a sub-linear mapping S.

In the following, X and Z are real vector space. The organization of this paper is as follows. In the next section, we list some definitions and lemmas. We present the Hahn-Banach theorem where the order vector space Z is induced by S-convex in Section 3.

2. PRELIMINARIES:

For later discussion, some definitions and lemmas are introduced.

Definition: 2.1 Let Z be a nontrivial vector space. We say that $S : Z \to \mathbb{R}$ is sub-linear if

$$x, y \in Z \Rightarrow S(x+y) \le S(x) + S(y)$$

and

$$x \in Z \text{ and } \lambda > 0 \Rightarrow S(\lambda x) = \lambda S(x)$$

In order to obtain the order in general vector space Z, we make use of S-convex. In the following, we give the definition of S-convex function.

Definition: 2.2 Let Z be a nontrivial vector space and $S: Z \to \mathbb{R}$ be sub-linear. Let K be a nonempty convex subset of a vector space and $g: K \to Z$. We say that g is S-convex if, for all $x \in Z$,

$$x_1, x_2 \in Z, \mu_1, \mu_2 > 0 \text{ and } \mu_1 + \mu_2 = 1$$

$$\Rightarrow S(g(\sum_i \mu_i x_i)) \le S(\sum_i \mu_i g(x_i))$$

Note that if we define an ordering "$\le_S$" on Z by declaring that $y \le_S z$ if, for all $x \in Z$, $S(y) \le S(z)$, then g is S-convex if and only if,

$$x_1, x_2 \in Z, \mu_1, \mu_2 > 0 \text{ and } \mu_1 + \mu_2 = 1$$

$$\Rightarrow g(\sum_i \mu_i x_i)) \le_S \sum_i \mu_i g(x_i)$$

An affine function is clearly S-convex. We denote the algebraic interior of $A \subset X$ by A^i. Recall that for A a convex set, one has

$$a \in A^i \Leftrightarrow \forall x \in X, \exists \lambda \in \mathrm{P}: a + \lambda x \in A$$

where $\mathrm{P} := (0, \infty) \subset \mathbb{R}$. We denote the space of linear operators from X into Z by $L(X, Z)$.

3. THE HAHN-BANACH THEOREM:

We state the classical Hahn-Banach theorem firstly. It says that a linear functional defined on a subspace X_0 of a real vector space X and which is dominated by a sub-linear functional defined on X has a linear extension which is also dominated by the same sub-linear functional.

Theorem: 3.1 Let X be a real vector space and p be a sub-linear functional defined on X. Suppose X_0 is a subspace of X and T_0 is a linear functional defined on X_0 such that $T_0(x) \le p(x)$ for all $x \in X_0$. Then there exists a linear functional T defined on X such that $T(x) = T_0(x)$ for all $x \in X_0$ and $T(x) \le p(x)$ for all $x \in X$.

Usual proof of this theorem involves two steps (See [3] for details). Of course, if $X_0 = X$, there is nothing to prove. When X_0 is a proper subspace, some $\bar{x} \in X \setminus X_0$ is chosen and T_0 can be extended to the linear span X_1 of $X \bigcup \{\bar{x}\}$ in such a way that the extension T remains dominated by p in X_1. This is the first step. The crucial idea in this step is to choose the value of $T(\bar{x})$ in an appropriate manner. The second step is to make use of Zorn's Lemma to construct a maximal subspace containing X_0 to which T_0 can be extended satisfying the required condition. Finally it is shown using the first step that this maximal subspace must coincide with X.

In the following section, in the vector space Z, we define an ordering "$\le_S$" on Z by declaring that $y \le_S z$ if, for all $x \in Z, S(y) \le S(z)$. And (Z, S) has the least upper bounded property, i.e. every non-empty and upper bounded set has a least upper bound, of course, in the situation, every non-empty and lower bounded set has a greatest lower bound. If the non-empty set $B \subset Z$ is bounded from below (above), we denote by $\inf B (\sup B)$ the greatest lower (least upper bound of B; if B is not bounded from below (above), we take $\inf B := -\infty (\sup B := +\infty)$. In addition, we also take the notation $\inf B := +\infty$ and $\sup B := -\infty$.

We give the main results in the following:

Theorem: 3.2 Let X be a vector space, $X_0 \subset X$ is linear subspace of X, and Z be a vector space, $S: Z \to \mathbb{R}$ be sub-linear, $g: X \to Z$ is S-convex, and $domg$ is convex, $T_0: X_0 \to Z$ is linear which is defined on X_0 such that

$$T_0(x) \le_S g(x), \forall x \in X_0 \bigcap domg.$$

Suppose that $domg^i \bigcap X_0 \neq \varnothing$, then there exists $T \in L(X,Z)$ such that

$$T\big|_{X_0} = T_0$$

$$T(x) \leq_S g(x), \forall x \in: X.$$

Proof: The theorem holds trivially if $X_0 = X$. Assume that $X_0 \neq X$. Since X_0 is a proper subspace of X , there exists $\bar{x} \in X \setminus X_0$. Let $X_1 = X_0 + \mathbb{R}\bar{x}$. It is clear that X_1 is a subspace of X , $X_0 \subset X_1$, $X_1 \bigcap domg^i \neq \varnothing$; and the above representation of $x_1 = x_0 + r\bar{x}$ is unique.

Note that it is sufficient to prove that, if $\bar{x} \in X \setminus X_0$, then there exists $T_1 : X_1 \to Z$, with $X_1 = X_0 + \mathbb{R}\bar{x}$ such that $T_1\big|_{X_1} = T_0$, and $T_1(x) \leq_S g(x), \forall x \in: X_1 \bigcap domg$. If so, by virtue of Zorn's lemma, as in the standard proof of the Hahn-Banach theorem, we obtains a maximal T defined on X .

Let $\bar{x}$ and X_1 be defined as above. Note that for each point $x \in X_0 \bigcap domg^i$, if $|k|$ is sufficiently small, then $x \pm k\bar{x} \in X_1 \bigcap domg$. Thus for arbitrary $x_1, x_2 \in X_0 \bigcap domg^i$, if

$x_1 + \lambda\bar{x}$, $x_2 - \mu\bar{x} \in X_1 \bigcap domg$ and $\lambda > 0$, $\mu > 0$, then set $\alpha = \dfrac{\lambda}{\lambda + \mu}$, and $\beta = 1 - \alpha = \dfrac{\mu}{\lambda + \mu}$.

It follows that

$$\alpha(x_1 + \lambda\bar{x}) + \beta(x_2 - \mu\bar{x}) = (\alpha x_1 + \beta x_2) + \alpha\lambda\bar{x} - \beta\mu\bar{x}$$

Since $\alpha\lambda\bar{x} - \beta\mu\bar{x} = 0$, then

$$\alpha(x_1 + \lambda\bar{x}) + \beta(x_2 - \mu\bar{x}) = \alpha x_1 + \beta x_2 \in domg.$$

Since $x_1, x_2 \in X_0$, and X_0 is linear subspace of X , so $\alpha x_1 + \beta x_2 \in X_0$. It follows that

$$\alpha x_1 + \beta x_2 \in X_0 \bigcap domg \, .$$

Since g is S -convex, and $T_0(x) \leq_S g(x), \forall x \in X_0 \bigcap domg$, then

$$\begin{aligned} T_0(\alpha x_1 + \beta x_2) &\leq_S g(\alpha x_1 + \beta x_2) \\ &= g(\alpha(x_1 + \lambda\bar{x}) + \beta(x_2 - \mu\bar{x})) \\ &\leq_S \alpha g(x_1 + \lambda\bar{x}) + \beta g(x_2 - \mu\bar{x})) \end{aligned}$$

Since T_0 is linear, so $T_0(\alpha x_1 + \beta x_2) = \alpha T_0(x_1) + \beta T_0(x_2)$. It follows that

$$\alpha T_0(x_1) + \beta T_0(x_2) \leq_S \alpha g(x_1 + \lambda\bar{x}) + \beta g(x_2 - \mu\bar{x}))$$

It is easy to obtain that

$$\beta T_0(x_2) - \beta g(x_2 - \mu\bar{x})) \leq_S \alpha g(x_1 + \lambda\bar{x}) - \alpha T_0(x_1)$$

Dividing $\alpha\beta$ by two sides in the above inequality, it follows that

$$\frac{T_0(x_2) - g(x_2 - \mu\bar{x}))}{\alpha} \leq_S \frac{g(x_1 + \lambda\bar{x}) - T_0(x_1)}{\beta}$$

By the definition of α and β , then we obtain that

$$\frac{T_0(x_2) - g(x_2 - \mu\bar{x}))}{\mu} \leq_S \frac{g(x_1 + \lambda\bar{x}) - T_0(x_1)}{\lambda}$$

Set the sets

$$B_1 = \{\frac{T_0(x) - g(x - \mu\bar{x}))}{\mu}\Big| x \in X_0, \mu > 0, x - \mu\bar{x} \in domg\}$$

$$B_2 = \{\frac{g(x + \lambda\bar{x})) - T_0(x)}{\lambda}\Big| x \in X_0, \lambda > 0, x + \lambda\bar{x} \in domg\}$$

and it follows that B_1 is bounded from above, B_2 is bounded from below, and $supB_1 \leq_S infB_2$ .Taking $\bar{y} \in Z$ such that $supB_1 \leq_S \bar{y} \leq_S infB_2$. Then we have

$$\frac{T_0(x) - g(x - \mu\bar{x})}{\mu} \leq_S \bar{y} \leq_S \frac{g(x + \lambda\bar{x}) - T_0(x)}{\lambda}$$

It follows that

$$T_0(x) - g(x - \mu\bar{x}) \leq_S \mu\bar{y} \tag{1}$$

$$g(x + \lambda\bar{x}) - T_0(x) \geq_S \lambda\bar{y} \tag{2}$$

Take $\mu = -\lambda$, then by (1),

$$T_0(x) - g(x + \lambda\bar{x}) \leq_S -\lambda\bar{y}$$

i.e.,

$$g(x + \lambda\bar{x}) - T_0(x) \geq_S \lambda\bar{y}$$

Define

$$T_1(x + \lambda\bar{x}) = T_0(x) + \lambda\bar{y}, \forall x \in X_0, \lambda \in \mathbb{R} .$$

When $\lambda = 0$, $T_0(x) = T_1(x), \forall x \in X_0$. We also have that $T_1 : X \to Z$ is linear, and $T_1(x) \leq_S g(x), \forall x \in X_1$. Indeed, if $\lambda < 0$, then $\lambda_1 = -\lambda > 0$ and $T_1(x) = T_0(x) - \lambda_1\bar{y}$. Take $b_1 := \frac{T_0(x) - g(x + \lambda_1\bar{x})}{\lambda_1}$, we have that $b_1 \in B_1$. and

$$\begin{aligned} T_1(x) \leq_S g(x), \forall x \in X_1 &\Leftrightarrow T_1(x + \lambda\bar{x}) \leq_S g(x + \lambda\bar{x}), \forall x \in X_0, \lambda < 0 \\ &\Leftrightarrow T_0(x) - \lambda_1\bar{y} \leq_S g(x - \lambda_1\bar{x}), \forall x \in X_0, \lambda_1 > 0 \\ &\Leftrightarrow T_0(x) - g(x - \lambda_1\bar{x}) \leq_S \lambda_1\bar{y}, \forall x \in X_0, \lambda_1 > 0 \\ &\Leftrightarrow \frac{T_0(x) - g(x - \lambda_1\bar{x})}{\lambda_1} \leq_S \bar{y}, \forall x \in X_0, \lambda_1 > 0 \\ &\Leftrightarrow b_1 \leq_S \bar{y} \end{aligned}$$

If $\lambda > 0$, then the proof is similarly. Let Γ be the collection of all ordered pairs (X_Δ, T_Δ) where X_Δ is a subspace of X that contains X_0 and T_Δ is a linear map from X_Δ to Z that extends T_0 and satisfies $T_\Delta \leq_S g(x), \forall x \in X_\Delta$.

Introduce a partial ordering in Γ as follows: $(X_{\Delta_1}, T_{\Delta_1}) \prec (X_{\Delta_2}, T_{\Delta_2})$ if and only if $X_{\Delta_1} \subset X_{\Delta_2}$, $T_{\Delta_1} = T_{\Delta_2}$ for all $x \in X_{\Delta_1}$. If we can show that every totally ordered subset of Γ has an upper bound, it will follow from Zorn's Lemma that Γ has a maximal element $(X_{\max}, T_{\max})$ .We can claim that $T_{\max}$ is the desired map. In fact, we must have $X_{\max} = X$. Otherwise, we have shown in the previous proof of this theorem that there would be an $(\tilde{X}_{\max}, \tilde{T}_{\max}) \in \Gamma$ such that $(\tilde{X}_{\max}, \tilde{T}_{\max}) \succ (X_{\max}, T_{\max})$ and $(\tilde{X}_{\max}, \tilde{T}_{\max}) \neq (X_{\max}, T_{\max})$. This would be violate the maximality of the $(X_{\max}, T_{\max})$. Therefore, it remains to show that every totally ordered subset of Γ has an upper bound. Let M be a totally ordered subset of Γ .

Define an ordered pair (X_M, T_M) by $X_M = \cup_{(X_\Delta, T_\Delta) \in M}\{X_\Delta\}$. And $T_M(x) = T_\Delta(x), \forall x \in X_\Delta$, where $(X_\Delta, T_\Delta) \in M$.

This definition is well-defined. Since if $(X_{\Delta_1}, T_{\Delta_1})$ and $(X_{\Delta_2}, T_{\Delta_2})$ are any elements of M , then either $(X_{\Delta_1}, T_{\Delta_1}) \prec (X_{\Delta_2}, T_{\Delta_2})$ or $(X_{\Delta_1}, T_{\Delta_1}) \succ (X_{\Delta_2}, T_{\Delta_2})$. In other words, if $x \in X_{\Delta_1} \cap X_{\Delta_2}$, then $T_{\Delta_1}(x) = T_{\Delta_2}(x)$ Clearly, $(X_M, T_M) \in \Gamma$. It follows that, it is an upper bound for M . The proof is complete.

Remark: 3.1 Theorem 3.2 generalizes the results in Theorem 3.1. In particular, the space $\mathbb{R}$ in Theorem 3.1 is generalized to general vector space Z whose order is induced by sub-linear mapping. Theorem 3.2 says that if a linear operator defined on a subspace X_0 of a real vector space X is dominated by a S -convex operator defined on X , then it has a linear extension which is also dominated by the same S -convex operator defined on X .

Remark: 3.2 The S -convexity of g is new in Theorem 3.2. Theorem 3.2 generalizes the sub-linear functional to general operator which is S -convex. Obviously, S -convex is weaker than convexity. Meanwhile, if g is sub-linear, then it is S -convexity. Thus, view in this point, our results generalizes the corresponding section in Theorem 3.1.

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