Research Journal of Pure Algebra -1(9), 2011, Page: 268-273
Available online through www.rjpa.info ISSN 2248–9037

GENERAL RELATIONS FOR ABSOLUTE SUMMABILITY

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(Received on: 22-09-11; Accepted on: 06-10-11)

ABSTRACT

New general results concerning absolute summability of an infinite series are presented. Other special cases are also deduced.

2000 Mathematics subject classification: *40F05, 40D25.*

Key words: *absolute summability, summability factor.*

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1. INTRODUCTION:

Let T be a lower triangular matrix, (s_n) a sequence, then

$$T_n := \sum_{v=0}^{n} t_{nv} s_v \tag{1}$$

A series $\sum a_n$ is said to be summable $|T|_k, k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta T_{n-1}|^k < \infty \tag{2}$$

Given any lower triangular matrix T one can associate the matrices $\overline{T}$ and T, with entries defined by,

$$t_{nv} = \sum_{i=v}^{n} t_{ni}, \quad n, i = 0, 1, 2, \ldots.., \quad \hat{t}_{nv} = \overline{t}_{nv} - \overline{t}_{n-1,v}$$

respectively. With $s_n = \sum_{i=0}^{n} a_i \lambda_i$,

$$t_n = \sum_{v=0}^{n} t_{nv} s_v = \sum_{v=0}^{n} t_{nv} \sum_{i=0}^{v} a_i \lambda_i = \sum_{i=0}^{n} a_i \lambda_i \sum_{v=i}^{n} t_{nv} = \sum_{i=0}^{n} \overline{t_{ni}} a_i \lambda_i \tag{3}$$

$$Y_n := t_n - t_{n-1} = \sum_{i=0}^{n} \overline{t}_n a_i \lambda_i - \sum_{i=0}^{n-1} \overline{t}_{n-1,i} a_i \lambda_i = \left| \sum_{i=0}^{n} \overline{t}_{ni} a_i \lambda_i \right| \ as \ \overline{t}_{n-1,n} = 0 \tag{4}$$

$$X_n = u_n - u_{n-1} = \sum_{i=0}^{n} u_{ni} a_i. \tag{5}$$

We call T a triangle if T is lower triangular and $t_{nn} \neq 0$ for all n. A triangle A is called factorable if its nonzero entries a_{mn} can be written in the form $b_m c_n$ for each m and n. We also assume that $U = (u_{ij})$ is a triangle $(p_n), (q_n)$ are assumed to be positive sequences of numbers such that

$$P_n = p_0 + p_1 + \ldots + p_n \to \infty, as \quad n \to \infty,$$

$$Q_n = q_0 + q_1 + \ldots + q_n \to \infty, as \quad n \to \infty,$$

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The series $\sum a_n$ is said to be summable $\left|R, p_n\right|_k, k \geq 1,$ if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\Delta z_{n-1}\right|^k < \infty,$$

where

$$z_n = \sum_{i=0}^{n} p_i s_i .$$

Via giving another name of this kind of summability (that is replacing $\left|R, p_n\right|_k$ by $\left|\overline{N}, p_n\right|_k$ with exactly the same meaning, Savas [1] gave the following result

Theorem: 1.1 *Let* $1 < k \leq s < \infty.$ *Let* (λ_n) *be a sequence of constants, T be a triangle with bounded entries such that T and* (p_n) *satisfy*

(i) $t_{vv}\lambda_v = \mathrm{O}\left(\left(\frac{p_v}{P_v}\right) v^{1/s-1/k}\right).$

(ii) $\left(n\left|X_n\right|\right)^{s-k} = \mathrm{O}(1),$

(iii) $\sum_{v=1}^{n-1}\left|\Delta_v\left(t_{nv}\lambda_v\right)\right| = \mathrm{O}\left(\left|t_{nn}\lambda_n\right|\right),$

(iv) $\sum_{n=v+1}^{\infty}\left(n\left|t_{nn\lambda_n}\right|\right)^{s-1}\left|\Delta_v\left(\hat{t}_{nv}\lambda_v\right)\right| = \mathrm{O}\left(v^{s-1}\left|t_{vv}\lambda_v\right|^s\right),$

(v) $\sum_{v=1}^{n-1}\left|t_{vv}\lambda_v\right|\left|\hat{t}_{nv}\lambda_v\right| = \mathrm{O}\left(\left|t_{nn}\lambda_n\right|\right),$ and

(vi) $\sum_{n=v+1}^{\infty}\left(n\left|t_{nn}\lambda_n\right|\right)^{s-1}\left|\hat{t}_{nv}\lambda_v\right| = \mathrm{O}\left(\left(v\left|t_{vv}\lambda_v\right|\right)^{s-1}\right),$

where $\left(X_n = u_n - u_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v, u_n = \frac{1}{P_n}\sum_{i=0}^{n} p_i s_i\right)$

Then the series $\sum a_n\lambda_n$ is summable $\left|T\right|_k$ whenever $\sum a_n$ is summable $\left|\overline{N}, p_n\right|_k$.

2. RESULT:

The aim of this paper is to present the following general result:

Theorem: 2.1 Let $1 < k \leq s < \infty, (\lambda_n)$ be a sequence of constants. Let T and U be triangles with bounded entries such that *U is* factorable, that is u_{nv} can be written as $u_{nv} = \phi_n\varphi_v,$ and they satisfy the following:

(i) $t_{vv}\lambda_v = \mathrm{O}\left(v^{1/s-1/k}\left|\phi_v\varphi_{v+1}\right|\right),$

(ii) $\left(n\left|X_n\right|\right)^{s-k} = \mathrm{O}(1),$

(iii) $\sum_{v=1}^{n-1}\left|\Delta_v\left(\hat{t}_{nv}\lambda_v\right)\right| = \mathrm{O}\left(\left|t_{nn}\lambda_n\right|\right),$

(iv) $\sum_{n=v+1}^{\infty}\left(n\left|t_{nn}\lambda_n\right|\right)^{s-1}\left|\Delta_v\left(\hat{t}_{nv}\lambda_v\right)\right| = \mathrm{O}\left(v^{s-1}\left|t_{vv}\lambda_v\right|^s\right),$

(v) $\sum_{v=1}^{n-1}|t_{vv}\lambda_v||\hat{t}_{nv}\lambda_v| = \mathrm{O}(|t_{nn}\lambda_n|)$,

(vi) $\sum_{n=v+1}^{\infty}(n|t_{nn}\lambda_n|)^{s-1}|\hat{t}_{nv}\lambda_v| = \mathrm{O}((v|t_{vv}\lambda_v|)^{s-1})$, and

(vii) $\Delta\varphi_v^{-1} = \mathrm{O}(|\phi_v|)$.

Then the series $\sum a_n\lambda_n$ is summable $|T|_s$ whenever $\sum a_n$ is summable $|U|_k$. It may be mentioned that theorem 1.1 remains a special case of theorem 2.1 in the sense that

If we are putting $U \equiv (R, p_n)$, that is $U = (u_{nv})$, where $\hat{u}_{nv} = \varphi_n\phi_v$, $\phi_n = \dfrac{p_n}{P_nP_{n-1}}, \varphi_v = p_{v-1}$ in theorem 2.1 we obtain theorem 1.1.

Proof: By Abel's transformation we have

$$\begin{aligned}
Y_n &= \sum_{v=1}^{n}\varphi_v a_v\,\frac{\hat{t}_{nv}\lambda_v}{\varphi_v} \\
&= \sum_{v=1}^{n-1}\left(\sum_{r=1}^{v}\varphi_r a_r\right)\Delta_v\left(\frac{\hat{t}_{nv}\lambda_v}{\varphi_v}\right) + \left(\sum_{v=1}^{n}\varphi_v a_v\right)\frac{t_{nv}\lambda_n}{\varphi_n} \\
&= \sum_{v=1}^{n-1}\frac{X_v}{\varphi_v}\left(\Delta\varphi^{-1}t_{nv}\lambda_v + \varphi_{v+1}^{-1}\Delta_v\left(\hat{t}_{nv}\lambda_v\right)\right) + \frac{X_n t_{nn}\lambda_n}{\phi_n\varphi_n} \\
&= Y_{n1} + Y_{n2} + Y_{n3}.
\end{aligned}$$

To complete the proof, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{s-1}|Y_{nj}|^s < \infty,\ j = 1, 2, 3.$$

Now applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{s-1}|Y_{n1}|^s &= \sum_{n=1}^{\infty} n^{s-1}\left|\sum_{v=1}^{n-1}\Delta\left(\varphi_v^{-1}\right)\frac{X_v\hat{t}_{nv}\lambda_v}{\varphi_v}|t_{vv}\lambda_v|^{-1}|t_{vv}\lambda_v|\right| \\
&\le \sum_{n=1}^{\infty} n^{s-1}\sum_{v=1}^{n-1}\left|\Delta\left(\varphi_v^{-1}\right)\right|^s\frac{|\hat{t}_{nv}\lambda_v||X_v|^s|t_{vv}\lambda_v|^{1-s}}{|\phi_v|^s}\left(\sum_{v=0}^{n-1}|t_{vv}\lambda_v||\hat{t}_{nv}\lambda_v|\right)^{s-1} \\
&= \mathrm{O}(1)\sum_{n=1}^{\infty} n^{s-1}|t_{nn}\lambda_n|^{s-1}\sum_{v=1}^{n-1}\left|\Delta\left(\varphi_v^{-1}\right)\right|^s\frac{|X_v|^s|t_{vv}\lambda_v|^{1-s}|\hat{t}_{nv}\lambda_v|}{|\phi_v|^s} \\
&= \mathrm{O}(1)\sum_{v=1}^{\infty}\left|\Delta\left(\varphi_v^{-1}\right)\right|^s\frac{|X_v|^s|t_{vv}\lambda_v|^{1-s}}{|\phi_v|^s}\sum_{n=v+1}^{\infty} n^{s-1}|t_{nn}\lambda_n|^{s-1}|\hat{t}_{nv}\lambda_n| \\
&= \mathrm{O}(1)\sum_{v=1}^{\infty}\left|\Delta\left(\varphi_v^{-1}\right)\right|^s\frac{|X_v|^s|t_{vv}\lambda_v|^{1-s}}{|\phi_v|^s}(v|t_{vv}\lambda_v|)^{s-1} \\
&= \mathrm{O}(1)\sum_{v=1}^{\infty} v^{s-1}|X_v|^s
\end{aligned}$$

$$= O(1)\sum_{v=1}^{\infty} v^{k-1}|X_v|^k \left(v|X_v|\right)^{s-k}$$

$$= O(1)\sum_{v=1}^{\infty} v^{s-1}|X_v|^k = O(1).$$

$$\sum_{n=1}^{\infty} n^{s-1}|Y_{n2}|^s = \sum_{n=1}^{\infty} n^{s-1}\left|\sum_{v=1}^{n-1}\frac{\Delta_v(t_{nv}\lambda_v)}{\phi_v\varphi_{v+1}}X_v\right|^s$$

$$\leq \sum_{n=1}^{\infty} n^{s-1}\left(\sum_{v=1}^{n-1}\frac{|\Delta_v(t_{nv}\lambda_v)|}{|\phi_v||\varphi_{v+1}|}|X_v|\right)^s$$

$$\leq \sum_{n=1}^{\infty} n^{s-1}\sum_{v=1}^{n-1}\frac{|\Delta_v(t_{nv}\lambda_v)||X_v|^s}{|\phi_v|^s|\varphi_{v+1}|^s}\left(\sum_{v=1}^{n-1}|\Delta_v(t_{nv}\lambda_v)|\right)^{s-1}$$

$$= O(1)\sum_{n=1}^{\infty} n^{s-1}|t_{nn}\lambda_n|^{s-1}\sum_{v=0}^{n-1}\frac{|\Delta_v(t_{nv}\lambda_v)||X_v|^s}{|\phi_v|^s|\varphi_{v+1}|^s}$$

$$= O(1)\sum_{v=1}^{\infty}\frac{|X_v|^s}{|\phi_v|^s|\varphi_{v+1}|^s}\sum_{n=v+1}^{\infty} n^{s-1}|t_{nn}\lambda_n|^{s-1}|\Delta_v(t_{nv}\lambda_v)|$$

$$= O(1)\sum_{v=1}^{\infty}\frac{v^{s-1}|t_{vv}\lambda_v|^s|X_v|^s}{|\phi_v|^s|\varphi_{v+1}|^s}$$

$$= O(1)\sum_{v=1}^{\infty} v^{s-s/k}|X_v|^s$$

$$= O(1)\sum_{v=1}^{\infty} v^{k-1}|X_v|^k\left(v^{s-s/k-k+1}|X_v|^{s-k}\right)$$

$$= O(1)\sum_{v=1}^{\infty} v^{k-1}|X_v|^k = O(1)$$

$$\sum_{n=1}^{\infty} n^{s-1}|Y_{n3}|^s = \sum_{n=1}^{\infty} n^{s-1}\left|\frac{t_{nn}X_n\lambda_n}{\phi_n\varphi_n}\right|^s$$

$$\leq \sum_{n=1}^{\infty} n^{s-1}\frac{|t_{nn}\lambda_n|^s|X_n|^s}{|\phi_n|^s|\varphi_n|^s}$$

$$= O(1)\sum_{n=1}^{\infty} n^{s-1}|X_n|^s$$

$$= O(1)\sum_{n=1}^{\infty} n^{k-1}|X_n|^k = O(1).$$

Theorem: 2.2 *Let* $1<k<\infty, (\lambda_n)$ *be a sequence of constants. Let T and U be triangles with bounded entries such that* $\hat{U}$ *is factorable, that is* $\hat{u}_{nv}$ *can be written as* $\hat{u}_{nv}=\phi_n\varphi_v,$ *and they satisfy the following:*

(i) $t_{vv}\lambda_v = O(|\phi_v\varphi_{v+1}|).$

(ii) $\sum_{v=1}^{n-1}\left|\Delta_v(\hat{t}_{nv}\lambda_v)\right| = O(|t_{nn}\lambda_n|),$

(iii) $\sum_{n=v+1}^{\infty}\left(n\left|t_{nn}\lambda_n\right|\right)^{k-1}\left|\Delta\left(\hat{t}_{nv}\lambda_v\right)\right| = \mathrm{O}\left(v^{k-1}\left|t_{vv}\lambda_v\right|^k\right)$

(iv) $\sum_{v=1}^{n-1}\left|t_{vv}\lambda_v\right|\left|\hat{t}_{nv}\lambda_v\right| = \mathrm{O}\left(\left|t_{nn}\lambda_n\right|\right)$, and

(v) $\sum_{n=v+1}^{\infty}\left(n\left|t_{nn}\lambda_n\right|\right)^{k-1}\left|\hat{t}_{nv}\lambda_v\right| = \mathrm{O}\left(\left(v\left|t_{vv}\lambda_v\right|\right)\right)$, and

(vi) $\Delta\varphi_v^{-1} = \mathrm{O}\left(\left|\phi_v\right|\right)$.

Then the series $\sum a_n\lambda_n$ is summable $\left|T\right|_k$ whenever $\sum a_n$ is summable $\left|U\right|_k$.

Proof: Follows from Theorem 2.1 by putting $s = k$.

Corollary: 2.3 *Sufficient conditions for the series* $\sum a_n$ *to be summable* $\left|R, q_n\right|_k$, $k > 1$, *whenever it is summable* $\left|R, p_n\right|_k$ *are*

(i) $q_nP_n = \mathrm{O}\left(p_nQ_n\right)$,

(ii) $\sum_{n=v+1}^{\infty}\frac{n^{s-1}q_n^s}{Q^sQ_{n-1}} = \mathrm{O}\left(\frac{v^{s-1}q_v^{s-1}}{Q_v^s}\right)$.

Proof: Follows from theorem 2.2.by putting.

$$\hat{t}_{nv} = \frac{q_nQ_{v-1}}{Q_nQ_{n-1}},\ \hat{u}_{nv} = \frac{p_nP_{v-1}}{P_nP_{n-1}},\quad \phi_n = \frac{p_n}{P_nP_{n-1}},\quad \varphi_v = P_{v-1},\quad and\ \lambda_n = 1,$$

noticing that $\hat{t}_{vv} = t_{vv}$, $\hat{u}_{vv} = u_{vv}$ for all v. We will show that conditions of Theorem 2.2 are all satisfied.

(1) By (i) $t_{vv} = \frac{q_v}{Q_v} = \mathrm{O}\left(\frac{p_v}{P_v}\right) = \mathrm{O}\left(\frac{p_vP_v}{P_vP_{v-1}}\right) = \mathrm{O}\left(\left|\phi_v\varphi_{v+1}\right|\right)$.

(2) $\sum_{v=1}^{n-1}\left|\Delta_v\hat{t}_{nv}\right| = \sum_{v=1}^{n-1}\frac{q_nq_v}{Q_nQ_{n-1}} = \frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n-1}q_v = \frac{q_n}{Q_n} = \mathrm{O}\left(\left|t_{nn}\right|\right)$.

(3) By (ii) $\sum_{n=v+1}^{\infty}\left(n\left|t_{nn}\right|\right)^{k-1}\left|\Delta_v\hat{t}_{nv}\right| = \sum_{n=v+1}^{\infty}\left(n\frac{q_n}{Q_n}\right)^{k-1}\frac{q_nq_v}{Q_nQ_{n-1}} = q_v\sum_{n=v+1}^{\infty}\frac{n^{k-1}q_n^k}{Q_n^kQ_{n-1}}$

$$= \mathrm{O}\left(\frac{v^{k-1}q_v^k}{Q_v^k}\right) = \mathrm{O}\left(v^{k-1}\left|t_{vv}\right|^k\right).$$

(4) $\sum_{v=1}^{n-1}\left|t_{vv}\right|\left|\hat{t}_{nv}\right| = \sum_{v=1}^{n-1}\frac{q_v}{Q_v}\frac{q_nQ_{v-1}}{Q_nQ_{n-1}} = \mathrm{O}\left(\frac{q_n}{Q_nQ_{n-1}}\right)\sum_{v=1}^{n-1}q_v = \mathrm{O}\left(\frac{q_n}{Q_n}\right) = \mathrm{O}\left(\left|t_{nn}\right|\right)$.

(5) $\sum_{n=v+1}^{\infty}\left(n\left|t_{nn}\right|\right)^{k-1}\left|\hat{t}_{nv}\right| = \sum_{n=v+1}^{\infty}n^{k-1}\left(\frac{q_n}{Q_n}\right)^{k-1}\frac{q_nQ_{v-1}}{Q_nQ_{n-1}} = Q_{v-1}\left|\sum_{n=v+1}^{\infty}\frac{n^{k-1}q_n^k}{Q_n^kQ_{n-1}}\right|$

$$= \mathrm{O}\left(\frac{v^{k-1}q_v^{k-1}}{Q_v^{k-1}}\right), by(ii)$$

$$= \mathrm{O}\left(\left(v\left|t_{vv}\right|\right)^{k-1}\right).$$

Corollary: 2.4 *The series* $\sum a_n$ *is summable* $\left|R, q_n\right|_k$ *iff it is summable* $\left|R, q_n\right|_k$. $k > 1$, *provided the following holds*

(i) $q_n P_n = \mathrm{O}\left(p_n Q_n\right)$,

(ii) $p_n Q_n = \mathrm{O}\left(q_n P_n\right)$,

(iii) $\sum_{n=v+1}^{\infty} \frac{n^{s-1} q_n^s}{Q_n^s Q_{n-1}} = \mathrm{O}\left(\frac{v^{s-1} q_v^{s-1}}{Q_v^s}\right)$,

(iv) $\sum_{n=v+1}^{\infty} \frac{n^{s-1} p_n^s}{P_n^s P_{n-1}} = \mathrm{O}\left(\frac{v^{s-1} p_v^{s-1}}{p_v^s}\right)$,

Proof: Follows from Corollary 2.3, via changing the rolls of $\left(p_n\right), \left(q_n\right)$.

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