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Properties of ℓ-module

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ABSTRACT

In this paper , we obtained some properties of ℓ -module over a ring R, prove the relation between ℓ - module and Browerian algebra and also established that ℓ -module is a direct product of Browerian algebra and ℓ - group.

Mathematical Subjectclass[2000]: *{06F25,06F99}*

1. INTRODUCTION AND PRELIMINARIES:

The relation between logics of algebra and modern algebra was worked by many mathematicians from Boolean. But in 1935 it was M.H. Stone who emphasized that there is a precise connection between lattice ordered groups and Boolean Algebra. He observed that a distributive complemented lattice is a Boolean ring with identity.

In connection with the work of M. H. Stone, Birkhoff, G.,imposed an open problem 105 in [1]: Is there a common abstraction which includes Boolean algebras (Rings) and lattice ordered groups as special cases. To answer this problem by the way of introducing the algebraic structure connecting lattice and group is called ℓ- group or lattice ordered group etc. Many propreties connected with dually residuated lattice ordered semigroups was presented in [2].

Definition 1.1 *Let R be a ring with unit element.A non-empty set M is said to be a ℓ -module over R, if it is equipped with the binary operations $+$, scalar multiplication and binary relation $\leq$ defined on it and satisfy the following conditions*
(i) M is a module over a ring R.
(ii) $(M,\leq)$ is a lattice.
(iii) $x \leq y \Rightarrow a+x \leq a+y$, for all $a, x, y \in M$
(iv) $x \leq y \Rightarrow \alpha x \leq \alpha y$, for all $x, y \in M$ and $\alpha \in R$,with $\alpha > 0, \alpha$ is unit.

Definition 1.2 *Let R be a ring with unit element.A non-empty set M is said to be a ℓ -module over R,if it is equipped with binary operations $+$, scalar multiplication, $\vee$ and $\wedge$ defined on it and satisfy the following conditions*
(i) M is a module over a ring R.
(ii) $(M,\vee,\wedge)$ is a lattice.
(iii) $a+(x \vee y) = (a+x) \vee (a+y)$
(iv) $a+(x \wedge y) = (a+x) \wedge (a+y)$,for all $a, x, y \in M$
(v) $\alpha(x \vee y) = \alpha x \vee \alpha y$, $\alpha(x \wedge y) = \alpha x \wedge \alpha y$
for all $x, y \in M$ and $\alpha \in R$ with $\alpha > 0, \alpha$ is unit

2. PROPERTIES:

Property: 2.1 *If M is a ℓ module then,* $[(a-b) \vee 0]+b = a \vee b$ *,for all a,b in M .*

Property: 2.2 *If M is a ℓ module then, $a \leq b \Rightarrow a-c \leq b-c$ and $c-b \leq c-a$ for all a,b,c in M*

Property: 2.3 *If M is a ℓ module then,* $(a \vee b)-c = (a-c) \vee (b-c)$ *, for all a,b,c in M .*

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Property: 2.4 *If* M *is a* ℓ *module then,* $a-(b\vee c)=(a-b)\wedge(a-c)$, *for all* a,b,c *in* M

Property: 2.5 *If* M *is a* ℓ *module then,* $a-(b\wedge c)=(a-b)\vee(a-c)$, *for all* a,b,c *in* M

Property: 2.6 *If* M *is a* ℓ *module then,* $(b\wedge c)-a=(b-a)\wedge(c-a)$, *for all* a,b,c *in* M

Property: 2.7 *If* M *is a* ℓ *module then,* $a\geq b\Rightarrow(a-b)+b=a$, *for all* a,b *in* M .

Property: 2.8 *If* M *is a* ℓ *module then,* $(a\vee b)+(a\wedge b)=a+b$, *for all* a,b *in* M .

Property: 2.9 *If* M *is a* ℓ *module then,* $[(a-b)\vee 0]+a\wedge b=a$, *for all* a,b *in* M

Property: 2.10 *If* M *is a* ℓ *module then,* $(a\vee b)-(a\wedge b)=(a-b)\vee(b-a)$,
for all a,b *in* M .

Property: 2.11 *In a* ℓ *module* M *then,* $a-(b-c)\leq(a-b)+c$ *and* $(a+b)-c\leq(a-c)+b$, for all a,b,c in M .

Property: 2.12 *In a* ℓ*- module* M *if* $a\wedge b=0$ *and* $a\wedge c=0$, then $a\wedge(b+c)=0$, if $a\vee b=0$ and $a\vee c=0$, then $a\vee(b+c)=0$, for all $a,b,c\in M$

Theorem: 2.1 *Any* ℓ*-module* M *is a distributive lattice.*

Proof: Given M is a ℓ-module.

To prove that M is a distributive lattice.

It is sufficient to prove
$a\vee x=a\vee y, a\wedge x=a\wedge y$ implies $x=y$, for all a,x,y in M .

Let a,x,y in M be arbitrary.

Suppose $a\vee x=a\vee y, a\wedge x=a\wedge y$

We have $a+b-a\wedge b=a\vee b$, *by property* 2.8

Putting $b=x$, in the above eqn, we get

$$\begin{aligned}
&a+x-a\wedge x=a\vee x\\
&\Rightarrow x=a\vee x+a\wedge x-a\\
&=a\vee y+a\wedge y-a\\
&x=a+y-a, \text{ by property } 2.8\\
&\Rightarrow x=y
\end{aligned}$$

Thus $a\vee x=a\vee y, a\wedge x=a\wedge y$
$\Rightarrow x=y$, for all a,x,y in M .

Hence M is a distributive lattice.

Theorem: 2.2 *Any Browerian algebra* B *is a* ℓ*-module over* B .

Proof: Given B is a Browerian Algebra.

To prove B is a ℓ-module over B .

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It is sufficient to prove.

(1) $(B,+,s.m)$ is a module.

(2) $x \leq y$ implies $a+x \leq a+y$ for all $a,x,y \in B$

(3) $x \leq y \Rightarrow \alpha x \leq \alpha y$, for all $x,y \in B, \alpha \in B, \alpha > 0, \alpha$ is unit.

Define $+, \cdot =$ scalar multiplication on B, by

$x+y = x \vee y, xy = x \wedge y$, for all $x,y \in B$

Scalar multiplication is closed

Let $x \in B, \alpha \in B$ be arbitrary, where α is unit $\alpha > 0$

$\Rightarrow \alpha \wedge x \in B \Rightarrow \alpha x \in B$

Next to claim

(i) $\alpha(x+y) = \alpha x + \alpha y$

(ii) $(\alpha + \beta)x = \alpha x + \beta x$

(iii) $(\alpha\beta)x = \alpha(\beta x)$

(iv) $1 \cdot x = x$

for all $\alpha, \beta, x, y \in B$

Therefore B is a module over B.

For(2): Let $a,x,y \in B$ be arbitrary.

Suppose $x \leq y \Rightarrow x \vee y = y$ and $x = x \wedge y$. Then

$$
\begin{aligned}
a+x = a+y \\
&= a+(x \vee y) \\
&= (a+x) \vee (a+y)
\end{aligned}
$$

Thus $x \leq y \Rightarrow a+x \leq a+y$, for all $a,x,y \in B$

For (3): Let $\alpha, x, y \in B$ be arbitrary , where $\alpha > 0, \alpha$ is unit

Suppose $x \leq y$

$\Rightarrow x \vee y = y$

$\Rightarrow \alpha(x \vee y) = \alpha y$

$\Rightarrow \alpha x \vee \alpha y = \alpha y$

$\Rightarrow \alpha x \leq \alpha y$, for all $\alpha, x, y \in B$

Therefore the Browerian algebra B is a ℓ - module.

Theorem: 2.3 *Any ℓ - module M is a direct product of a Browerian Algebra B and an ℓ -group G if and only if*

$(i)(a+b)-(c+c) \geq (a-c)+(b-c)$ and

$(ii)(ma+nb)-(a+b) \geq (ma-a)+(nb-b)$

for all $a,b,c \in M$ and any two positive integers m,n.

Proof: To prove B is a Browerian Algebra, G is a ℓ - group and $M = B \times G$

Take $B = \{a \in M / a+a-a = 0\}$

Claim: 1 B is a Browerian Algebra.

Claim: 2 G is a ℓ - group.

(ii) $(G, \vee, \wedge)$ is a lattice.

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Next to claim that

1. $a+(x\vee y)=(a+x)\vee(a+y)$
2. $a+(x\wedge y)=(a+x)\wedge(a+y)$, for all $a,x,y\in G$

Hence G is a ℓ - group.

Claim: 3 $M=B\times G$

It is enough to prove for any $a\in M$ can be uniquely expressed as $a=x+y$,
$y=(a+a)-a, x=a-[(a+a)-a]\Rightarrow x\in B$ and $y\in G$ .Then

$$
\begin{aligned}
(y+y)-y &= [(2a-a)+(2a-a)]-(2a-a) \\
&= [(2a+2a)-(a+a)]-(2a-a),\ by\ (3.19) \\
&= (4a-2a)-(2a-a) \\
&\geq 4a-2a-a,\ since\ 2a-a\leq a \\
&= a=(a+a)-a=y
\end{aligned}
$$

$\Rightarrow (y+y)-y\geq y$

Also $(y+y)-y\leq(y-y)+y,\ by\ property\ 3.11$
$= y$
$\Rightarrow (y+y)-y\leq y$

Therefore $(y+y)-y=y$

$\Rightarrow y\in G$
$y=(a+a)-a$
$\Rightarrow y\leq a$

$x\geq 0\Rightarrow x+x\geq 0+x=x$
$\Rightarrow x+x\geq x$

Now, $(a-y)+(a-y)=(a+a)-(y+y),\ by\ (3.17)$

$$
\begin{aligned}
&= 2a-2y \\
&= 2a-2(2a-a) \\
&= 2a-(4a-2a)
\end{aligned}
$$

$\Rightarrow x+x=2a-(4a-2a)$

We have

$$
\begin{aligned}
(4a-2a)+(a-(2a-a)) &= (2a-a)+(2a-a)+(a-(2a-a)) \\
&\geq (2a-a)+(2a-a)+0,\ since\ x\geq 0
\end{aligned}
$$

$\Rightarrow (4a-2a)+(a-(2a-a))\geq 2a$
$\Rightarrow 2a-(4a-2a)\leq a-(2a-a)$
$\Rightarrow x+x\leq x$
Therefore $x+x=x$
$\Rightarrow x+x-x=0$
$\Rightarrow x\in B$

Thus if $a\in M$ then $a=x+y$ implies $x\in B, y\in G$

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Uniqueness part: Suppose $a = x' + y'$ where $x' \in B$ and $y' \in G$
Hence $M = B \times G$

Second Part: Conversely assume that a ℓ -module $M = B \times G$ where B is a Browerian algebra and G is a ℓ - group.

To prove
(i) $(a+b)-(c+c) \geq (a-c)+(b-c)$
(ii) $(ma+nb)-(a+b) \geq (ma-a)+(nb-b)$, for all $a,b,c \in M$ and any two positive integers m,n .

Let $a,b,c \in M$ be arbitrary.
$\Rightarrow a,b,c \in B$,since $a = a+0, b = b+0, c = c+0$
$\Rightarrow a-c, b-c, a+b \in B$
$\Rightarrow (a-c)+(b-c), a+b \in B$
$\Rightarrow (a-c)+(b-c)-(a+b) \in B$ such that

$(a+b) \vee x \geq (a-c)+(b-c),$ since B is a Browerian algebra

$\Rightarrow x = -(c+c) \in B$ such that
$(a+b)-(c+c) \geq (a-c)+(b-c)$

Similarly let $a,b \in M$ be arbitrary.
$\Rightarrow a,b \in B$
$\Rightarrow ma, nb,, a, b \in B,$ since $a+a = a, a+a+a = a$ etc.
$\Rightarrow ma-a, nb-b, ma+nb \in B$
$\Rightarrow (ma-a)+(nb-b), ma+nb \in B$
$\Rightarrow x = [(ma-a)+(nb-b)]-(ma+nb)$ B such that
$\Rightarrow (ma+nb)+x \geq (ma-a)+(nb-b)$
$\Rightarrow (ma+nb)-(a+b) \geq (ma-a)+(nb-b)$

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