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PRIME AVOIDANCE THEOREM FOR CO-IDEALS IN SEMIRINGS

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ABSTRACT

In this paper we characterize subtractive co-ideals, prime co-ideals in the semiring $(Z^+, +, \cdot)$ *and obtain prime avoidance theorem for co-ideals in semirings.*

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1. INTRODUCTION:

A non-empty set R together with two associative binary operations addition and multiplication is called a semiring if i) addition is a commutative operation ii) there exists $0 \in R$ such that $x + 0 = x = 0 + x$, $x \cdot 0 = 0 = 0 \cdot x$ for each $x \in R$ and iii) multiplication distributes over addition both from left and right. The concept of commutative semiring, semiring with identity element, ideal can be defined on the similar lines as in rings. All semirings in this paper are assumed to be commutative with identity element. Z^+ (N) will denote the set of all non-negative (positive) integers.

Definition: 1.1 ([4]) A non-empty subset I of a semiring R is called a co-ideal if

1. $a, b \in I$ implies $ab \in I$;
2. $a \in I$, $r \in R$ implies $a + r \in I$.

Definition: 1.2 A co-ideal I of a semiring R is called subtractive if
$a, ab \in I$, $b \in R$, then $b \in I$.

Definition: 1.3. A proper co-ideal I of a semiring R is called prime if
$a + b \in I$, then $a \in I$ or $b \in I$.

2. Co-ideals in the semiring Z^+

For $n \in Z^+$, we denote $I_n = \{a \in Z^+ : a \geq n\}$.

Lemma: 2.1 A non-empty subset I of the semiring $(Z^+, +, \cdot)$ is a co-ideal if and only if $I = I_n$ for some $n \in Z^+$.

Proof: Let I be a non-empty subset of Z^+. Therefore I has the smallest element say n. If I is a co-ideal of Z^+, then $x + n \in I$ for all $x \geq 0$. Hence $I_n \subseteq I$. But $I \subseteq I_n$. Hence $I = I_n$. Conversely, if $I = I_n$, then clearly I is a co-ideal of Z^+.

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Theorem: 2.2 A co-ideal I of the semiring $(Z^+, +, \cdot)$ is a subtractive co-ideal if and only if $I = I_1$ or $I = I_0$.

Proof: Let I be a subtractive co-ideal of Z^+. By Lemma 2.1, $I = I_n$ for some $n \in Z^+$. If $n \geq 2$, then n, $\mathbf{1} \cdot n \in I$ but $\mathbf{1} \notin I_n$, a contradiction to I is a subtractive co-ideal. Hence $n = \mathbf{1}$ or 0. Converse is trivial.

Theorem: 2.3 A co-ideal I of the semiring $(Z^+, +, \cdot)$ is a prime co-ideal if and only if $I = I_1$.

Proof: Let I be a prime co-ideal of Z^+. Therefore $I \neq Z^+ = I_0$. By Lemma 2.1, $I = I_n$ for some $n \in N$. If $n \geq 2$, then $(n-1)+\mathbf{1} = n \in I$. Since I is a prime co-ideal, either $n-1 \in I = I_n$ or $\mathbf{1} \in I = I_n$, a contradiction. Hence $n = \mathbf{1}$. Converse is trivial.

Theorem: 2.4 A non-empty subset I of the semiring $(Z^+ \cup \{\infty\}, \max, \min)$ is a co-ideal if and only if

$I = I_n$ for some $n \in Z^+$ or $I = Z^+ \cup \{\infty\}$ or $I = \{\infty\}$.

Proof: Let I be a co-ideal of $Z^+ \cup \{\infty\}$ and $I \neq Z^+ \cup \{\infty\}$, $I \neq \{\infty\}$. Choose the smallest element n such that $n \in I$. So $I \subseteq I_n$. Now if $x \in I_n$, then $x = \max\{x, n\} \in I$. So $I_n \subseteq I$. Hence $I = I_n$. Conversely, if $I = I_n$ for some $n \in Z^+$ or $I = Z^+ \cup \{\infty\}$ or $I = \{\infty\}$, then clearly I is a co-ideal.

3. Prime Avoidance Theorem for co-ideals in semirings:

N. Jacobson [5] has proved the Prime Avoidance Theorem for prime ideals in commutative rings. Further it is generalized by P. J. Allen, J. Neggers and H. Kim [2] for subtractive prime ideals in commutative semirings. We generalize this theorem for subtractive prime co-ideals in commutative semirings.

Theorem: 3.1 Let I, J be co-ideals of a semiring R. Then
(i) $I \cap J$ is a co-ideal of R; ii) $I + J$ is a co-ideal of R; iii) $I + J \subseteq I \cap J$;

Proof: (i) Trivial.

(ii) Let $x = a + b$, $y = a' + b' \in I + J$, $r \in R$ where a, $a' \in I$, b, $b' \in J$. Since I is a co-ideal, $aa' \in I$ and $aa' + ab' + ba' \in I$. Also $bb' \in J$ as J is a co-ideal. Now $xy = aa' + ab' + ba' + bb' \in I + J$. Since J is a co-ideal, $x + r = a + (b + r) \in I + J$. Hence $I + J$ is a co-ideal R.

(iii) Let $x = a + b \in I + J$ where $a \in I$, $b \in J$. Now $a \in I$, $b \in R$ and I is a co-ideal implies $x = a + b \in I$.

Similarly $x \in J$. Hence $x \in I \cap J$. Thus $I + J \subseteq I \cap J$.

The following example shows that union of two co-ideals of a semiring R is not a co-ideal.

Example: 3.2 Let $X = \{a, b, c\}$ and $R = (P(X), \cup, \cap)$ a semiring where $P(X)$ = the set of all subsets of X. Then clearly $I = \{X, \{a, b\}\}$ and $J = \{X, \{b, c\}\}$ are co-ideals of R. But $I \cup J = \{X, \{a, b\}, \{b, c\}\}$ is not a co-ideal because $\{a, b\} \cap \{b, c\} = \{b\} \notin I \cup J$.

For m, $n \in R = (Z^+, +, \cdot)$, it is easy to verify that

(i) $I_n \cap I_m = I_t$ where $t = \max\{n, m\}$;
(ii) $I_n \cup I_m = I_r$ where $r = \min\{n, m\}$;
(iii) $I_n + I_m = I_{n+m}$.

Inclusion in the Theorem 3.1 (iii) is strict because by Lemma 2.1, I_{10}, I_{15} are co-ideals in the semiring $(Z^+, +, \cdot)$ and $I_{10} + I_{15} = I_{25} \subset I_{10} = I_{10} \cap I_{15}$.

Theorem: 3.3 Let I, J be subtractive co-ideals of a semiring R. Then $I \cap J$ is a subtractive co-ideal of R.

Proof: By Theorem 3.1 (i), $I \cap J$ is a co-ideal of R. If a, $ab \in I \cap J$, then $b \in I \cap J$ as I, J are subtractive co-ideals of R.

Following example shows that sum of two subtractive co-ideals is not a subtractive co-ideal.

Example: 3.4 By Theorem 2.2, I_1 is a subtractive co-ideal in the semiring $(Z^+, +, \cdot)$. But $I_1 + I_1 = I_2$ is not a subtractive co-ideal.

Lemma: 3.5 Let I, J be subtractive co-ideals of a semiring R and $a \in I$, $b \in J$. Then the following conditions are equivalent:
1. $ab \in I \cup J$;
2. $a \in I \cap J$ or $b \in I \cap J$;
3. $a, b \in I$ or $a, b \in J$.

Proof: (1) $\Rightarrow$ (2) Let $ab \in I \cup J$. Without loss of generality assume that $ab \in I$. Since I is a subtractive co-ideal, we have $b \in I$. Hence $b \in I \cap J$. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are trivial.

Theorem: 3.6 Let I, J be subtractive co-ideals of a semiring R and A be a co-ideal of R. Then $A \subseteq I \cup J$ if and only if $A \subseteq I$ or $A \subseteq J$.

Proof: Let $A \subseteq I \cup J$, $A \neq I$ and $A \not\subset I$. Let $a \in A$ be such that $a \notin I$. Then $a \in J$. We claim that $A \cap I \subseteq J$. Let $b \in A \cap I$. Since A is a co-ideal, $ab \in A \subseteq I \cup J$. By Lemma 3.5, $a, b \in I$ or $a, b \in J$. Then $a, b \in J$, since $a \notin I$. Hence $A \cap I \subseteq J$. Now $A = A \cap (I \cup J) = (A \cap I) \cup (A \cap J) \subseteq J$. Converse is trivial.

Corollary: 3.7 Let I, J be subtractive co-ideals of a semiring R. Then $I \cup J$ is a co-ideal of R if and only if $I \subseteq J$ or $J \subseteq I$.

Theorem: 3.8 (Prime Avoidance Theorem) Let $I_1, I_2, I_3, \ldots, I_n$ be subtractive co-ideals of a semiring R such that at most two of the I_r are not prime. If I is a co-ideal of R such that $I \subseteq I_1 \cup I_2 \cup I_3 \cup \ldots \cup I_n$, then $I \subseteq I_r$ for some r.

Proof: Proof is by induction on n. If $n = 1$, then the result is trivial. Let n = 2. Suppose that $I \subseteq I_1 \cup I_2$. By Theorem 3.6, $I \subseteq I_1$ or $I \subseteq I_2$. Assume the result is true for $n-1$, $n \geq 3$. Let $I_1, I_2, I_3, \ldots, I_n$ be subtractive co-ideals of a semiring R such that at most two of the I_r's are not prime and $I \subseteq I_1 \cup I_2 \cup I_3 \cup \ldots \cup I_n$, $n \geq 3$. Suppose that I is not a subset of union of any (n - 1) I_r's. Choose $a_j \in I$ such that $a_j \notin I_r$ for all $r \neq j$ ... (*). Clearly $a_j \in I_j$. Since $n \geq 3$, without loss of generality assume that I_1 is a prime co-ideal.

Now
$a_1(a_2 + a_3 + \ldots + a_n) \in I \subseteq I_1 \cup I_2 \cup I_3 \cup \ldots \cup I_n$, since I is a co-ideal of R. Therefore $a_1(a_2 + a_3 + \ldots + a_n) \in I_1$ or $a_1(a_2 + a_3 + \ldots + a_n) \in I_r$ for some r, $2 \leq r \leq n$.

Case: (i) $a_1(a_2 + a_3 + \ldots + a_n) \in I_1$. Then $a_2 + a_3 + \ldots + a_n \in I_1$ as I_1 is a subtractive co-ideal. Since I_1 is a prime co-ideal, $a_j \in I_1$ for some $2 \leq j \leq n$, a contradiction to (*).

Case: (ii) $a_1(a_2 + a_3 + \ldots + a_n) \in I_r$ for some r, $2 \leq r \leq n$. Since each I_r is a subtractive co-ideal and $a_2 + a_3 + \ldots + a_n \in I_r$, $a_1 \in I_r$, a contradiction to (*). Thus in each case we get a contradiction. Hence I is a subset of union of some (n - 1) I_r's. So by induction assumption I $\subseteq I_r$ for some r.

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