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σ – PROJECTIVITY AND σ – SEMI-SIMPLICITY IN MODULES

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ABSTRACT

An exact sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is called $\mathcal{T}$ −pure ($\mathfrak{F}$- copure) if any torsion (torsion free) $R-$ module is projective (injective) relative to it. Since $\mathcal{T}(\mathfrak{F})$ is closed under factors (sub-modules). In this situation Walker's [23] criterion of Co-purity is also applicable. The notation of an $R-$ module M is $\mathcal{T}$ −pure projective ($\mathfrak{F}$- copure injective) if and only if $Pext_{\mathcal{T}}(M, A) = 0$ ($Pext_{\mathfrak{F}}(A, M) = 0$) for all $A \subseteq M$. In particular $Pext_{\mathcal{T}}(T, A) = 0$ for all $T \in \mathcal{T}$.We denote the torsion sub-module of $A \subseteq M$ by $\sigma(A)$. Walker proved that the class of $\mathcal{I}-$pure ($\mathfrak{F}-$ copure) sequences form a proper class whenever $\mathcal{I}(\mathfrak{F})$ is closed under homomorphic images (sub-modules) of an $R-$ module M and if $\mathcal{I}(\mathfrak{F})$ is closed under factors (sub-modules) then for any $\mathcal{I}-$pure ($\mathfrak{F}-$ copure) sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ if $E \in \pi^{-1}(\mathcal{I})$ ($E \in i^{-1}(\mathfrak{F})$) and hence in this case Walker's $\mathcal{I}-$purity ($\mathfrak{F}-$ copurity) coincides with the earlier notion of purity. We try to define a class of modules projective with respect to a torsion theory and to show that they are none other than $\mathcal{T}$ −pure flat modules. Here we define two torsion theoretic generalizations of projective modules and one of them will be characterized as $\mathcal{I}-$ pure flat modules. Also the $\sigma-$semisimple ring of Rubin [21] will be characterized in terms of divisibility and $\mathcal{I}-$ purity. We also study about divisible modules and co-divisible modules. we try to specify $\mathcal{T}$ −pure injective and $\mathcal{T}$ −pure projective modules and also we enumerate some properties of divisibility and co-divisibility as such to giving of characterization for exactness of a torsion theory in terms of it divisible and co-divisible $R-$ modules. Most of these results of the theorem are proved by Lambek [17] for $\mathcal{J}_1-$ purity. In this present paper we try to relate the strongly $\sigma-$ projectivity, $\sigma-$ projective modules, torsion $\sigma-$ projective modules and also, $\mathcal{J}-$ pure flat module. we try to give the inter relationship between torsion modules, divisible modules, co-divisible modules and semi-simplicity of the modules for a hereditary torsion theory with radical σ.

Keywords: *R −modules, torsion modules, $\sigma-$ pure projective R −modules, $\sigma-$ pure injective R −modules, $\mathcal{I}-$ pure ($\mathfrak{F}-$ copure), $\mathcal{I}-$ pure flat modules, Divisible modules, co- divisible modules, absolutely $\mathcal{I}_1$- purity.*

Subject classification: *16D99.*

1. INTRODUCTION

The notion of purity plays a fundamental role in the theory of abelian groups as well as in module categories. We say that an $R-$ module M is absolutely pure, (respectively regular, flat) with respect to the purity if any short exact sequence with M as the first (respectively second, third) position is pure in the given sense. An exact sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is called $\mathcal{T}$ −pure ($\mathfrak{F}$- copure) if any torsion (torsion free)$R-$ module is projective (injective) relative to it. Since $\mathcal{T}(\mathfrak{F})$ is closed under factors (sub-modules). In this situation Walker's [23] criterion of Co-purity is also applicable. The notation of an $R-$ module M is $\mathcal{T}$ −pure projective ($\mathfrak{F}$- copure injective) if and only if $Pext_{\mathcal{T}}(M, A) = 0$ ($Pext_{\mathfrak{F}}(A, M) = 0$) for all $A \subseteq M$. In particular $Pext_{\mathcal{T}}(T, A) = 0$ for all $T \in \mathcal{T}$.We denote the torsion sub-module of $A \subseteq M$ by $\sigma(A)$. Walker proved that the class of $\mathcal{I}-$pure ($\mathfrak{F}-$ copure) sequences form a proper class whenever $\mathcal{I}(\mathfrak{F})$ is closed under homomorphic images (sub-modules) of an $R-$ module M and if $\mathcal{I}(\mathfrak{F})$ is closed under factors (sub-modules) then for any $\mathcal{I}-$ pure ($\mathfrak{F}-$ copure) sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$

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if $E \in \pi^{-1}(\mathcal{I})$ $(E \in i^{-1}(\mathfrak{F}))$ and hence in this case Walker's $\mathcal{I}-$purity ($\mathfrak{F}-$ copurity) coincides with the earlier notion of purity. Here we define two torsion theoretic generalizations of projective modules and one of them will be characterized as $\mathcal{I}-$ pure flat modules. Also the $\sigma-$semisimple ring of Rubin [21] will be characterized in terms of divisibility and $\mathcal{I}-$ purity. We also study about divisible modules and co-divisible modules. we try to specify $\mathcal{T}-$pure injective and $\mathcal{T}-$pure projective modules and also we enumerate some properties of divisibility and co-divisibility as such to giving of characterization for exactness of a torsion theory in terms of it divisible and co-divisible $R-$ modules. Most of these results of the theorem are proved by Lambek [17] for $\mathcal{J}_1-$ purity. An $R-$module P is said to be $\sigma-$ projective if given an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ and a homomorphism $f: P \longrightarrow C$, then there exists a homomorphism $g: \sigma(P) \longrightarrow B$ such that $f|\,\sigma(P) = \lambda o g$, where $\lambda: B \longrightarrow C$. An $R-$module P is said to be strongly $\sigma-$ projective if given a homomorphism $f: P \longrightarrow C$, then there exists a homomorphism $g: P \longrightarrow B$ such that $f|\,\sigma(P) = \lambda o g|\sigma(P)$, where $\lambda: B \longrightarrow C$. There is a given torsion theory $(\mathcal{I}, \mathfrak{F})$ with radical σ, an $R-$ module M is called $\sigma-$ semi-simple if each dense sub-module N of M is a direct summand. This definition was given by Rubin [21]. We have already known that absolute $\mathcal{J}_1$- purity coincides with absolute $\mathcal{J}$- purity which is the case of divisibility in $R-$ modules.An exact sequence E is called $\mathcal{T}-$pure ($\mathfrak{F}$- copure) if any torsion (torsion free) module is projective (injective) relative to it. Since $\mathcal{T}(\mathfrak{F})$ is closed under factors (sub-modules). We know that an $R-$ module M is said to be divisible with respect to a torsion theory if it is injective relative to any exact sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with C torsion. Also, an $R-$module M is said to be co-divisible if M is $\mathfrak{F}-$copure flat module. We also study about divisible modules and co-divisible modules. we try to specify $\mathcal{T}-$pure injective and $\mathcal{T}-$pure projective modules and also we enumerate some properties of divisibility and co-divisibility as such to giving of characterization for exactness of a torsion theory in terms of it divisible and co-divisible $R-$ modules. Most of these results of the theorem are proved by Lambek [17] for $\mathcal{J}_1-$ purity. In this present paper we try to relate the strongly $\sigma-$ projectivity, $\sigma-$ projective modules torsion $\sigma-$ projective modules and also, with $\mathcal{J}-$ pure flat modules. we try to give the inter relationship between torsion modules divisible modules and co-divisible modules and semi-simplicity of the modules for a hereditary torsion theory with radical σ. $\mathcal{J}_1$- purity has the interesting property that if $M \in \mathfrak{F}$, then $N \subseteq M$ is $\mathcal{J}_1-$pure if and only if $M/N \in \mathfrak{F}$. All torsion free modules are $\mathcal{J}_1$- pure flat. The converse of this theorem holds if $\sigma(R) = 0$. Stenstrom [19] prop. 6.23).This concept of purity of sub-modules of torsion free modules have been used in the study of torsion-free covers. (Teply [20]). Given any complete sub-category which is closed under sub-modules and injective hulls. That is a torsion-free class of a hereditary torsion theory. If the concept of purity for sub-objects of objects of this sub-category which is defined by the above property, then the sub-category of absolutely pure modules form an abelian category (Mitchell [18]). An absolutely $\mathcal{J}_1$- pure modules are precisely the divisible modules. We also get that the sub-category of torsion-free divisible modules is an abelian category (Lambek[17]). Here we give some definitions which are used or related to this present paper.

Definition:

1. An $R-$ module M is said to be cyclic if and only if there exists an element $m_0 \in M$ such that $M = Rm_0$.
2. An $R-$ module M is said to be finitely generated if and only if there exists a finite generating set X of M.
3. A left $R-$ module M is said to finitely co-generated if and only if for each set $\{U_i\,|i \in I\}$ of submodules U_i of M with $\cap_{i\in I} U_i = 0$, there exists a finite subset $\{U_i\,|i \in I_0\}$ that is $I_0 \subset I$ and I_0 is finite with $\cap_{i\in I} U_i = 0$. In other words we can say A module M is said to be finitely co-generated if it is co-generated by the family $\{E(S_{i\in I})\}$ finitely. That is $E(M) = \oplus_{i=1}^{n} E(S_i)$ where $S_{i\in I}$ simple modules are not necessarily non-isomorphic.
4. An $R-$ module M is said to be cocyclic if it is contained in $E(S)$ for some simple module S, where $E(S)$ is a family of co-generators for each R module M.
5. In the commutative diagram
$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ M & \longrightarrow & N \end{array}$$
Where $f: A \longrightarrow B$; $\varphi: M \longrightarrow N, \mu: A \longrightarrow M \; and \; g: B \longrightarrow N$ be maps. The pair (φ, g) is said to be the pushout of the pair (μ, f) if and only if for every pair (φ', g') with
$\varphi': M \longrightarrow X, g': B \longrightarrow X$ and $(\varphi' o\, \mu) = (g' o f)$, there exists a unique map $\sigma: N \longrightarrow X$ such that $(\sigma o g) = g'$.
6. The pair (ϕ, f) is said to be the pullback of the pair (ψ, g) if and only if for every pair (ϕ', f') with $\phi': Y \to M$, $f': Y \to B$ and $(\psi o \phi') = (g o f')$, there exists a unique map
$\tau: Y \to A$ such that $(f o \tau) = f' and\; (\phi o \tau) = \phi'$.
7. An $R-$ module M is said to be finitely presented if there is an exact sequence $M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$ where M_0 and M_1 are free modules with finite bases.
8. Let R be a ring and M is a left $R-$ module, then M is said to flat if for every exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N" \longrightarrow 0$ and the transformed sequence
$0 \longrightarrow M \otimes_R N' \longrightarrow M \otimes_R N \longrightarrow M \otimes_R N" \longrightarrow 0$ is exact.
A ring R is hereditary if and only if every ideal is a projective module.
9. If M be an $R-$module, the sum all simple submodules of M is called the **socle of M** and it is denoted by $s(M) = \{x \in M | Ann(x)$ is a finite intersection of maximal right ideals$\}$. That is if $x \in s(M)$, then xA is a direct sum of a finite number of simple modules.

10. A non- zero module S is said to be simple if it has on submodules other than $\{0\}$ and S. A module is said to be semi-simple if it is a sum of simple sub-modules.
11. A torsion theory is a pair $(\mathcal{J}, \mathfrak{F})$ of classes of modules satisfying:
    (i). $Hom(T, F) = 0, \forall\, T \in \mathcal{J}$ and $F \in \mathfrak{F}$
    (ii). $Hom(L, F) = 0, \forall\, F \in \mathfrak{F} \Rightarrow L \in \mathcal{J}$
    (iii). $Hom(T, N) = 0, \forall\, T \in \mathcal{J} \Rightarrow N \in \mathfrak{F}$
12. The classes $\mathfrak{F}$ and $\mathcal{J}$ are known as torsion free and torsion classes associated with a torsion theory$(\mathcal{J}, \mathfrak{F})$. A torsion theory $(\mathcal{J}, \mathfrak{F})$ is said to be hereditary if and only if $\mathcal{J}$ is closed under homomorphic images, direct sums, extensions and sub-modules. Similarly, $\mathfrak{F}$ is closed under submodules, direct products, extensions and injective envelipes.
13. A left $R - module$ P is said to be $\sigma -$ pure projective module if it is projective to relative to every $\sigma -$ pure epimorphism. That is given any $\sigma -$ pure exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ and a homomorphism $f: P \longrightarrow C$, there exists a map $h: P \longrightarrow B$ such that $poh = f$ where $p: B \longrightarrow C$ be an onto homomorphism.
14. A left $R - module$ Q is said to be finitely $\sigma -$ pure injective if it is $(\mathcal{FG}, \sigma)$- pure in every pure extension of Q. That is if $0 \longrightarrow Q \longrightarrow Q' \longrightarrow Q'' \longrightarrow 0$ is a pure exact sequence then it is $(\mathcal{FG}, \sigma)$- pure also.Similarly, Q is said to be cyclically $\sigma -$ pure injective if it is cyclically $\sigma -$ pure in every pure extension of it.
15. A sub-module A of an R-module B is called closed if $B|A$ is torsion free and it is called dense if $B|A$ is torsion. Any closed submodule A of an R-module B is $\mathcal{T}$ −pure.
16. Given a class of modules $\mathcal{J}(\mathfrak{F})$, a sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is called $\mathcal{J} -$ pure ($\mathfrak{F} -$ copure) if A is a direct summand of D whenever $A \leq D \leq B$ and $D|A \in \mathcal{J}$ (A|S is a direct summand of B|S whenever $S \leq A$ and $A|S \in \mathfrak{F}$).
17. Given a class of modules $\mathcal{J}(\mathscr{J})$, a sequence E is called $\mathcal{J} -$ pure ($\mathscr{J} -$ copure) if A is a direct summand of D whenever $A \leq D \leq B$ and $D|A \in \mathcal{J}$ (A|S is a direct summand of B|S whenever $S \leq A$ and $A|S \in \mathscr{J}$). Walker proved that the class of $\mathcal{J} -$ pure ($\mathscr{J} -$ copure) sequences form a proper class whenever $\mathcal{J}(\mathscr{J})$ is closed under homomorphic images (submodules) and if $\mathcal{J}(\mathscr{J})$ is closed under factors (submodles) then any $\mathcal{J} -$ pure ($\mathscr{J} -$ copure) sequence if $E \in \pi^{-1}(\mathcal{J})(E \in i^{-1}(\mathcal{J}))$ and hence in this case Walker's $\mathcal{J}$ −purity ($\mathscr{J} -$ copurity) and hence in this case Walker's $\mathcal{J} -$ purity ($\mathscr{J} -$ copurity) coincides with the earlier notion.
18. A sub-module A of B is called closed if $B|A$ is torsion free and it is called dense if $B|A$ is torsion. Any closed submodule is $\mathcal{T}$ −pure.
19. Given a torsion theory $(\mathcal{T}, \mathfrak{F})$, an exact sequence E is called $\mathcal{T}$ −pure ($\mathfrak{F}$- copure) if any torsion (torsion free) module is projective (injective) relative to it. Since $\mathcal{T}(\mathfrak{F})$ is closed under factors (sub-modules), Walker's criterion of Co-purity is applicable. In this notation a module M is $\mathcal{T}$ −pure projective ($\mathfrak{F}$- copure injective) if and only if $Pext_{\mathcal{T}}(M, A) = 0$ $(Pext_{\mathfrak{F}}(A, M) = 0)$ for all $A \subseteq M$. In particular $Pext_{\mathcal{T}}(T, A) = 0$ for all $T \in \mathcal{T}$. We denote the torsion sub-module of A by $\sigma(A)$.

2. $\sigma -$ PROJECTIVITY AND $\sigma -$ SEMISIMPLICITY

We define two torsion theoretic generalizations of projective modules and one of them will be characterized as $\mathcal{J} -$ pure flat modules. Also the $\sigma -$ semisimple ring of Rubin [21] will be characterized in terms of divisibility and $\mathcal{J} -$ purity.

Definition 2.1: An R −module P is said to be $\sigma -$ projective if given an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ and a homomorphism $f: P \longrightarrow C$, then there exists a homomorphism $g: \sigma(P) \longrightarrow B$ such that $f|\,\sigma(P) = \lambda og$, where $\lambda: B \longrightarrow C$.

$$\begin{array}{ccccccccc} & & & & \sigma(P) & \rightarrow & P & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Definition 2.2: An R −module P is said to be strongly $\sigma -$ projective if given a homomorphism $f: P \longrightarrow C$, then there exists a homomorphism $g: P \longrightarrow B$ such that $f|\,\sigma(P) = \lambda og|\sigma(P)$, where $\lambda: B \longrightarrow C$.

Theorem 2.3:
(i) A strongly $\sigma -$ projective module is $\sigma -$ projective.
(ii) An $R -$ module P is $\sigma -$ projective if and only if given an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$, there exists $g: \sigma(P) \longrightarrow B$ such that $i_{\sigma(P)} = \lambda og$, where $\lambda: B \longrightarrow C$.
(iii) An R −module P is strongly $\sigma -$ projective if and only if given an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$, there exists $g: P \longrightarrow B$ such that $i_{\sigma(P)} = \lambda og|\sigma(P)$, where $\lambda: B \longrightarrow C$.
(iv) Every torsion $\sigma -$ projective module is projective.

Proof: **(i).** Trivial

(ii).
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & P_1 & \longrightarrow & P & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Given $f: P \longrightarrow C$, we can extend the above diagram by pullback. There exists a homomorphism $q: \sigma(P) \longrightarrow P_1$ such that $\pi o q = i_{\sigma(P)}$ then $\lambda(hoq) = f(\pi o q)h|\sigma(P)$ where $i: A \longrightarrow P_1$; $\pi: P_1 \longrightarrow P; j: A \longrightarrow B; \lambda: B \longrightarrow C$; $h: P_1 \longrightarrow B$ and $f: P \longrightarrow C$ are homomorphism. Converse of this part is obvious.
(iii). The proof of this part is similar as the proof of (ii).
(iv). It is trivial.

Theorem 2.4: If P is a $\sigma -$ Projective $R -$ module, then $\sigma(P)$ is a $\sigma(R)$- module. That is $\sigma(P)$ is a direct summand of a direct sum of copies of $\sigma(R)$.

Proof:

$$\begin{array}{ccccc} \sigma(\oplus R) = \oplus\, \sigma(R) & \rightleftarrows & \sigma(P) & & \\ \downarrow & \swarrow & & & \\ \oplus R & \rightarrow & P & \rightarrow & 0 \end{array}$$

Here, $\alpha': \oplus\, \sigma(R) \to \sigma(P)$; $\beta': \oplus\, \sigma(P) \to \sigma(R)$; $i: \oplus\, \sigma(R) \to \oplus R$;
$\beta: : \oplus\, \sigma(P) \to \oplus R$; $j: : \oplus\, \sigma(P) \to P$ and $\alpha: \oplus R \to P$ are homomorphism. If P is $\sigma -$ Projective $R -$ module and $\oplus R$ is the free $R -$ module generated over $P,$ then there exists a homomorphism $\beta: : \oplus\, \sigma(P) \to \oplus R$ such that $j = (\alpha o \beta)$. But we see that
$Im(\beta) \subseteq \sigma(\oplus R) = \oplus\, \sigma(R)$ and hence there exists a homomorphism $\beta': \oplus\, \sigma(P) \to \sigma(R)$ which satisfying $(io\beta') = \beta$. Now we have $jo(\alpha' o \beta') = \alpha o(io\beta) = (\alpha o \beta) = j$ and hence, $(\alpha' o \beta') = 1_{\sigma(P)}$ and $\sigma(P)$ is a direct summand of $\oplus\, \sigma(R)$. Hence, proved.

Theorem 2.5: An $R -$ module P is $\sigma -$ Projective if and only if it is a $\mathcal{J} -$ pure flat module.

Proof: Suppose that P is $\sigma -$ Projective $R -$ module. We consider an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$ and $f: T \longrightarrow P$ be a homomorphism where $T \in \mathcal{J}$. We have $f_1: T \longrightarrow \sigma(P)$; $f: T \longrightarrow P$,

$$\begin{array}{ccccccc} & & & & T & & \\ & & & \swarrow & & & \\ & & \sigma(P) & & \downarrow & & \\ & & \downarrow & \searrow & & & \\ 0 \longrightarrow & A & \longrightarrow & B & \longrightarrow & P & \longrightarrow 0 \end{array}$$

$f_2: \sigma(P) \longrightarrow P$; $g: \sigma(P) \longrightarrow B$ and $\lambda: B \longrightarrow P$ be homomorphisms. Now f factors through $\sigma(P)$. By the given hypothesis there exists $g: \sigma(P) \longrightarrow B$ such that $(\lambda o g) = f_2$. Now we see that $(\lambda o g) o f_1 = f_2 o f_1 = f$ and hence, the given sequence $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$ is $\mathcal{J} -$ pure. Thus P is $\mathcal{J} -$ pure flat module.

Conversely, If P is $\mathcal{J} -$ pure flat module, then given sequence $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$ is $\mathcal{J} -$ pure. Hence, there exists a homomorphism $g: \sigma(P) \longrightarrow B$ such that $(\lambda o g) = i_{\sigma(C)}$. Hence, P is $\sigma -$ projective R module by the using of the above theorem [2.3].

Proposition 2.6: The exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is $\mathcal{T} -$pure exact if and only if $0 \longrightarrow \sigma(A) \longrightarrow \sigma(B) \longrightarrow \sigma(C) \longrightarrow 0$ is a split exact sequence where the maps are restrictions of the above sequence.

Proof: Suppose that the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is $\mathcal{T} -$ pure exact. Now we complete the diagram by taking pullback of $j_C: \sigma(C) \longrightarrow C$ and $\pi: B \longrightarrow C$. Here,
$t: K \longrightarrow \sigma(B); u: \sigma(A) \longrightarrow \sigma(B); v: \sigma(B) \longrightarrow \sigma(C)$; $\alpha: \sigma(C) \to \sigma(B); s: \sigma(B) \to P$.

$$\begin{array}{ccccccccc} & & & & \mathrm{K} & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \sigma(A) & \longrightarrow & \sigma(B) & \longrightarrow & \sigma(C) & \longrightarrow 0 & (1) \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & P & \underset{\lambda}{\rightarrow} & \sigma(C) & \longrightarrow 0 & (2) \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 & (3) \end{array}$$

$q: P \to B$; $j_B: \sigma(B) \longrightarrow B, i': A \to P, \pi': P \to \sigma(C)$; $\lambda: \sigma(C) \to P, i: A \to B, \pi: B \to C$ are the required homomorphism.

Here s: $\sigma(B) \to P$ exists as P is a pullback. Put $K = \ker(v)$. Now $vou = 0$ and so, $\sigma(A) \subseteq K$. Since sequence (1) is $\mathcal{T}$ −pure $\Longrightarrow$ sequence (2) is $\mathcal{T}$ −pure because $\mathcal{T}$ −pure sequences form a proper class and hence (2) splits. Take
$\lambda: \sigma(C) \to P$ such that $\pi' o\, \lambda = 1_{\sigma(C)}$. Now $\lambda(\sigma(C))$ is torsion and so there is
$\alpha: \sigma(C) \to \sigma(B)$ such that $\lambda = so\alpha$. Also, $vo\alpha = \pi' o(so\alpha) = \pi' o\, \lambda = 1_{\sigma(C)}$ and hence, v is epic and $0 \longrightarrow K \rightleftarrows \sigma(B) \longrightarrow \sigma(C) \longrightarrow 0$ splits. But then K is an epimorphic image of $\sigma(B)$ and so, it is torsion. Also, $\pi' o(sot) = 0 \Longrightarrow K \subseteq A$. Hence, $K \subseteq \sigma(A)$ and sequence (3) is split and exact.

Conversely, if sequence (3) is split and exact, then given $T \in \mathcal{T}$, and $f: T \longrightarrow C$, $Im(f) \subseteq \sigma(C)$ and also, sequence (1) $\mathcal{T}$ −pure.

$$\begin{array}{cccccccccc} & & & & T & & & & & \\ & & & & \downarrow & & & & & \\ 0 & \longrightarrow & \sigma(A) & \longrightarrow & \sigma(B) & \rightleftarrows & \sigma(C) & \longrightarrow & 0 & \qquad (4) \\ & & \downarrow & & \downarrow & & \downarrow & & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 & \qquad (5) \end{array}$$

Note: If sequence (1) is $\mathcal{T}$ − pure, so it is exact on sequence (1) and hence, $\sigma(A) = A \cap \sigma(B)$ and $\frac{\sigma(B)+A}{A} = \sigma\left(\frac{B}{A}\right)$.

Theorem 2.7: A torsion theory $(\mathcal{J}, \mathfrak{F})$ is exact if and only if every torsion free $R -$ module M is divisible.

Proof: Suppose that each torsion free module is divisible. Let $T' \subseteq T$ and $T \in \mathcal{J}$, and let $F \in \mathfrak{F}$. Since, F is divisible, then any map $f: T' \longrightarrow F$ extends to a map $g: T \longrightarrow F$ and hence, $f = 0$ as $g = 0$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T' & \longrightarrow & T & \longrightarrow & T/T' & \longrightarrow & 0 \\ & & \downarrow & \swarrow & & & & & \\ & & F & & & & & & \end{array}$$

Hence, $T' \in \mathcal{J}$ and $(\mathcal{J}, \mathfrak{F})$ is a hereditary torsion theory. Now, let $C \in \mathfrak{F}$ and consider a factor $C"$ of C. We take a map $f: T \longrightarrow C"$ with $T \in \mathcal{J}$.Now $C \in \mathfrak{F} \Rightarrow C' \in \mathfrak{F}$ and hence C' is divisible and so the exact sequence $0 \longrightarrow C' \longrightarrow \longrightarrow C" \longrightarrow 0$ is $\mathcal{J}$- pure. Also,

$$\begin{array}{ccccccccc} & & & & & & T & & \\ & & & & & \swarrow & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C" & \longrightarrow & 0 \end{array}$$

Where there is a map $g: T \longrightarrow C$ which is the lifting of the map $f: T \longrightarrow C"$ and so, $g = 0$ as $C \in \mathfrak{F}$. Therefore, $f = 0$ and $C" \in \mathfrak{F}$. Hence, $(\mathcal{J}, \mathfrak{F})$ is a co-hereditary torsion theory also.

Conversely, suppose that the torsion theory $(\mathcal{J}, \mathfrak{F})$ is exact, then $\mathfrak{F}$ is closed under factors and injective hulls. Given $M \in \mathfrak{F}$ and $f: A \longrightarrow M$ be an $R -$ homomorphism, where A is dense in B, we extend the diagram by injective hull of M.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & E(M) & \longrightarrow & \dfrac{E(M)}{M} & \longrightarrow & 0 \end{array}$$

Where $f: A \longrightarrow M$; $\mu: B \longrightarrow M$; $\lambda: B \longrightarrow C$; $j: M \longrightarrow E(M)$; and $g: B \longrightarrow E(M)$; $h: C \longrightarrow \frac{E(M)}{M}$; $\pi: E(M) \longrightarrow \frac{E(M)}{M}$ are $R -$ homomorphisms. By hypothesis $E(M) \in \mathfrak{F}$ and $\frac{E(M)}{M} \in \mathfrak{F}$. So $h = 0$ and hence, $\pi o g = 0$. Thus there is a homomorphism $\mu: B \longrightarrow M$ such that $j o \mu = g$. But then $\mu o i = f$; $i: A \longrightarrow B$ and hence M is divisible.

Theorem 2.8: If a torsion module M is co-divisible if and only if every torsion module M having a projective cover in an exact torsion theory$(\mathcal{J}, \mathfrak{F})$ is co-divisible.

Proof: This follows dually of the proof of the above theorem.

Definition 2.9: There is a given torsion theory $(\mathcal{J}, \mathfrak{F})$ with radical σ, an $R -$ module M is called $\sigma -$ semi-simple if each dense submodule N of M is a direct summand. This definition was given by Rubin [21].

Theorem 2.10: The following statements are equivalent for a ring R:

(i) $R$is a $\sigma -$ semi-simple module.
(ii) Each R −module M is $\sigma -$ semisimple.
(iii) Each R −module M is $\sigma -$ projective.
(iv) Each exact sequence is $\mathcal{J} -$ pure.
(v) Each R −module M is divisible.
(vi) Each torsion R −module M is projective.
(Vii) Each dense ideal I is a direct summand of R.
(viii) Given a sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact, then the sequence
$0 \longrightarrow \sigma(A) \longrightarrow \sigma(B) \longrightarrow \sigma(C) \longrightarrow 0$ is also split exact sequence.
{ix} Every torsion module is semi-simple and σ is an exact functor.

Note: A ring R which satisfying these above conditions has been called σ −semi-simple Rubin[21].

Proof: $(i) \Leftrightarrow (vii)$ It is trivial.

$(ii) \Rightarrow (iii)$. Given any exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, we can extend it by taking pullback. Now A is dense in P and hence,

$$\begin{array}{ccccccccc} 0 \longrightarrow & A & \longrightarrow & P & \rightleftarrows & \sigma(C) & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

The upper sequence splits. Then $\pi o(qo\lambda') = io(\lambda o\lambda') = i$.

Where $\lambda: P \to \sigma(C)$, $\lambda': \sigma(C) \longrightarrow P$; $q: P \to B$; $i: (C) \to C$ and $\pi: B \to C$ are homomorphisms and hence, $C\ is\ \sigma$-projective.

$(iii) \Rightarrow (ii)$. Given any R −module B and any dense sub-module $A, B/A$ is torsion and since every $R-$ module M is $\sigma-$projective, B/A is projective and hence, A is a direct summand of B.Hence, every R −module M is σ −semisimple.

$(iii) \Leftrightarrow (iv) \Leftrightarrow (v)$. As we know that $\sigma -$ projectivity is equivalent to $\mathcal{I} -$ pure flatness, hence, the proof follows.

$(iii) \Rightarrow (vi)$. We know that each torsion σ- projective $R-$ module is projective.

$(vi) \Rightarrow (vii)$. This is trivial.

$(vii) \Rightarrow (v)$.The given exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & I & \longrightarrow & R & \longrightarrow & R/I & \longrightarrow 0 \\ & \downarrow & & & & & \\ & M & & & & & \end{array}$$

With the ideal I is dense in R any R −module M, R/I is projective and hence the above given sequence splits and M is injective relative to it. Thus the given $R-$ module is divisible.

$(iv) \Leftrightarrow (viii)$. The proof of this follows from the proposition [2.6].

$(viii) \Rightarrow (ix)$. If the statement $(viii)$ hold, then the radical σ is exact obviously. Moreover any $R-$ module M is σ −semisimple and thus any torsion module is semi-simple.

$(ix) \Rightarrow (viii)$. Given that the exact sequence

$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$………(1), firstly we have exactness of the exact sequence $0 \longrightarrow \sigma(N) \longrightarrow \sigma(B) \longrightarrow \sigma(C) \longrightarrow 0$……(2).

Since, here given as $\sigma(B)$ is a torsion module, so, it is semi-simple and hence, the above sequence (2) splits.

Proposition 2.11: The following statements are equivalent for a hereditary torsion theory with radical σ:
(i). Every torsion module is divisible.
(ii). Every torsion module is semi-simple.
(iii). $\sigma(M) \subseteq soc(M)$ for all R −module M.

Proof: $(i) \Rightarrow (ii)$. Given an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with $B \in \mathcal{I}$, we have A and$C \in \mathcal{I}$. Since A is divisible and $C \in \mathcal{I}$, the given sequence splits and hence B is semi-simple.

$(ii) \Rightarrow (i)$. Given an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with C torsion and any homomorphism$f: A \longrightarrow M$ with M is torsion module. Now we complete the diagram by pushout

$$\begin{array}{ccccccccc} 0 \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & M & \leftrightarrows & P & \longrightarrow & C & \longrightarrow 0 \end{array}$$

Now P is a torsion R −module as M and C are torsion R −modules. Hence, $P$is semi-simple and hence the lower sequence splits. Hence, M is divisible.

$(ii) \Leftrightarrow (iii)$. It is trivial.

Theorem 2.12: If $R \neq 0$ is a riing of $socle$ zero, then in the simple torsion theory of Dickson [14], there is a torsion module which is not co-divisible.

Proof: Suppose it is not then by theorem [2.7] and [2.8], the torsion theory is exact. Hence, in the simple torsion theory, every $R-$ module being a factor of a direct sum of copies of R, is torsion free. Hence, there is no simple module, which is impossible, because if there is no nonzero maximal ideal then there is no nonzero ideal and in this case 0 is a maximal ideal and Ritself is simple. But in this case $socle(R) = R$ and hence, R would be zero which is not in this case.

We have already known that absolute $\mathcal{J}_1$- purity coincides with absolute $\mathcal{J}$- purity which is the case of divisibility in $R-$ modules.

Proposition 2.13: All torsion free modules are $\mathcal{J}_1$- pure flat. The converse of this theorem holds if $\sigma(R) = 0$. (Stenstrom [19] prop. 6.23)

Remark:

1. $\mathcal{J}_1$- purity has the interesting property that if $M \in \mathfrak{F}$, then $N \subseteq M$ is $\mathcal{J}_1 -$ pure if and only if $M/N \in \mathfrak{F}$.This concept of purity of sub-modules of torsion-free modules have been used in the study of torsion-free covers. (Teply [20]).
2. Given any complete sub-category which is closed under sub-modules and injective hulls. That is a torsion-free class of a hereditary torsion theory. If the concept of purity for sub-objects of objects of this sub-category which is defined by the above property, then the sub-category of absolutely pure modules form an abelian category (Mitchell [18]). An absolutely $\mathcal{J}_1$- pure modules are precisely the divisible modules. We also get that the sub-category of torsion-free divisible modules is an abelian category (Lambek [17])

CONCLUSION

In this present paper we try to define a class of modules projective with respect to a torsion theory and to show that they are none other than $\mathcal{T}$ −pure flat modules. Here we define two torsion theoretic generalizations of projective modules and one of them will be characterized as $\mathcal{I} -$ pure flat modules and try to give the inter relationship between divisible modules and co-divisible modules. In this present paper we also try to relate the strongly $\sigma -$ projectivity, $\sigma -$ projective modules torsion $\sigma -$ projective modules and also, with $\mathcal{J} -$ pure flat modules. We try to give the inter relationship between torsion modules divisible modules and co-divisible modules and semi-simplicity of the modules for a hereditary torsion theory with radical σ. Most of these results of the theorem are proved by Lambek [17] for $\mathcal{J}_1 -$ purity.

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