Research Journal of Pure Algebra -1(8), 2011, Page: 196-204
Available online through www.rjpa.info ISSN 2248–9037

ON A PRODUCT SUMMABILITY OF AN INFINITE SERIES

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(Received on: 22-09-11; Accepted on: 24-11-11)

ABSTRACT

New results concerning product summability of an infinite series are given. Some special cases are also deduced.

1. INTRODUCTION:

Let $\sum a_n$ be a given infinite series with partial sums s_n. Let u_n^α denote the nth Cesaro mean of order $\alpha > -1$ of the sequence (s_n). The series $\sum a_n$ is summable

$|C, \alpha|_k, \; k \geq 1,$ if

(1) $$\sum_{n=1}^{\infty} n^{k-1} \left| u_n^\alpha - u_{n-1}^\alpha \right|^k < \infty \qquad (Flett\ [1]).$$

For $\alpha = 1$, $|C, \alpha|_k$ reduces to $|C, 1|_k$ summability.

Let (p_n) be a sequence of positive real constants such that $P_n = \ldots\ldots\ldots \to \infty$, as $n \to \infty$ $(P_{-1} = p_{-1} = 0)$. The (N, p) transform ϕ_n of (s_n) generated by (p_n) is defined by

(2) $$\phi_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v .$$

The sequence - to – sequence transformation

$$\Phi_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (Φ_n) of $(\overline{N}, p_n)$ transform of (s_n) generated by (p_n). The series $\sum a_n$ is summable $|R, p_n|_k, \; k \geq 1,$ if

(3) $$\sum_{n=1}^{\infty} n^{k-1} \left| \Phi_n - \Phi_{n-1} \right|^k < \infty .$$

In the special case when $p_n = 1$ for all n (resp. k=1), $|R, p_n|_k$ summability reduces to $|C, 1|_k$ (resp. $|R, p_n|$) summability.

The series $\sum a_n$ is said to be summable $|(N, p_n)(N, q_n)|$.....(N, p_n).....(N, q_n)..., when the (N, p) transform of the (N, q) transform of (s_n) is a sequence of bounded variation (see Das [2]).

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We give the following new definition:

Let (T_n) defines the sequence of the $(\overline{N}, q_n)$ transform of the $(\overline{N}, p_n)$ transform of (s_n) generated by the sequences (q_n) and (p_n) respectively. The series $\sum a_n$ is said to be summable $\left|(R, q_n)(R, p_n)\right|_k$, $k \geq 1$, *if*

(4) $$\sum_{n=1}^{\infty} n^{k-1} \left|T_n - T_{n-1}\right|^k < \infty.$$

We may assume through the paper that $Q_n = q_0 + ... + q_n \to \infty,\ as\, n \to \infty,\ q_n / Q_n \to 0,\ \ as\ \ n \to \infty.$

2. NEW RESULTS:

We state and prove the following

Theorem: 2.1 Let $k \geq 1$, (λ_n) be a sequence of constants. Define

$$f_v = \sum_{r=v}^{n} \frac{q_r}{P_r}, \quad F_v = \sum_{r=v}^{n} p_r f_r \ .$$

Let

(5) $$p_v Q_v = \mathrm{O}(P_v),$$

(6) $$P_v f_v \ = \mathrm{O}(v q_v),$$

(7) $$\sum_{n=v+1}^{\infty} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} = \mathrm{O}\left(\frac{(v q_v)^{k-1}}{Q_v^k}\right).$$

Then sufficient conditions for the implication

(8) $$\sum a_n \textit{ is summable } \left|R, q_n\right|_k \Rightarrow \sum a_n \lambda_n \textit{ is summable } \left|(R, q_n)(R, p_n)\right|_k$$

are

(9) $$\left|\lambda_n\right| < Q_n ,$$

(10) $$v p_v \left|\lambda_v\right| = \mathrm{O}(P_v),$$

(11) $$\left|\lambda_v\right| F_v = \mathrm{O}(Q_v),$$

(12) $$\left|\Delta \lambda_v\right| F_v = \mathrm{O}(q_v),$$

and

(13) $$\left|\Delta \lambda_v\right| = \mathrm{O}(q_v).$$

Proof: Let (S_n) be the sequence of partial sums of $\sum a_n \lambda_n$. Let v_n, V_n be the $(\overline{N}, q_n)$, $(\overline{N}, q_n)(\overline{N}, p_n)$ transforms of the sequences (s_n), (S_n) respectively. We write $t_n = v_n - v_{n-1}$, $T_n = V_n - V_{n-1}$. Therefore

(14) $$t_n = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v \ ,$$

and

$$V_n = \frac{1}{Q_n} \sum_{r=0}^{n} q_r \frac{1}{P_r} \sum_{v=0}^{r} p_v S_v$$

$$= \frac{1}{Q_n} \sum_{v=0}^{n} p_v S_v \sum_{r=v}^{n} \frac{q_r}{P_r}$$

$$= \frac{1}{Q_n} \sum_{v=0}^{n} p_v S_v f_v \ .$$

$$T_n = V_n - V_{n-1}$$

$$= \frac{q_n}{Q_n Q_{n-1}} \sum_{r=0}^{n} p_r S_r f_r + \frac{p_n S_n f_n}{Q_{n-1}}$$

$$= \frac{q_n}{Q_n Q_{n-1}} \sum_{r=0}^{v} p_r f_r \sum_{v=0}^{r} a_v \lambda_v + \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=0}^{n} a_v \lambda_v$$

(15) $$= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^{n} a_v \lambda_v \sum_{r=v}^{n} p_r f_r + \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=0}^{n} a_v \lambda_v$$

$$= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v \frac{\lambda_v}{Q_{v-1}} \sum_{r=v}^{n} p_r f_r + \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v \frac{\lambda_v}{Q_{v-1}}$$

$$= \frac{q_n}{Q_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(\sum_{r=1}^{v} Q_{r-1} a_r \right) \Delta_v \left(\frac{\lambda_v}{Q_{v-1}} \sum_{r=v}^{n} p_r f_r \right) + \left(\sum_{v=1}^{n} Q_{v-1} a_v \right) \frac{\lambda_n p_n f_n}{Q_{n-1}} \right)$$

$$+ \frac{p_n q_n}{P_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(\sum_{r=1}^{v} Q_{r-1} a_r \right) \Delta \left(\frac{\lambda_v}{Q_{v-1}} \right) + \left(\sum_{v=1}^{n} Q_{v-1} a_v \right) \frac{\lambda_n}{Q_{n-1}} \right)$$

$$= \frac{q_n}{Q_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(t_v \lambda_v F_v + \frac{Q_{v-1}}{q_v} p_v t_v \lambda_v f_v + \frac{Q_{v-1}}{q_v} t_v \Delta\lambda_v \, F_{v+1} \right) \right) + \frac{p_n}{Q_{n-1}} t_n \lambda_n f_n$$

$$+ \frac{p_n q_n}{P_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(t_v \lambda_v + \frac{Q_{v-1}}{q_v} t_v \Delta\lambda_v \right) \right) + \frac{p_n Q_n}{P_n Q_{n-1}} t_n \lambda_n ,$$

$$= \sum_{j=1}^{7} T_{nj} .$$

In order to complete the proof, it sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} \left| T_{nj} \right|^k < \infty, \quad j = 1,2,3,4,5,6,7.$$

Applying Hölder's inequality,

$$\sum_{n=2}^{m+1} n^{k-1} \left| T_{n1} \right|^k = \sum_{n=2}^{m+1} n^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} t_v \lambda_v F_v \right|^k$$

$$\le \sum_{n=2}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} \sum_{v=1}^{n-1} \frac{1}{q_v^{k-1}} \left| t_v \right|^k \left| \lambda_v \right|^k F_v^k \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^{k-1}$$

$$= O(1) \sum_{v=1}^{m} \frac{1}{q_v^{k-1}} \left| t_v \right|^k \left| \lambda_v \right|^k F_v^k \sum_{n=v+1}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} v^{k-1} \left| t_v \right|^k \frac{\left| \lambda_v \right|^k F_v^k}{Q_v^k} = O(1).$$

$$\sum_{n=2}^{m+1} n^{k-1} \left| T_{n2} \right|^k = \sum_{n=2}^{m+1} n^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{Q_{v-1} p_v}{q_v} t_v \lambda_v f_v \right|^k$$

$$\le \sum_{n=2}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} \sum_{v=1}^{n-1} \frac{Q_{v-1}^k p_v^k}{q_v^{2k-1}} \left| t_v \right|^k \left| \lambda_v \right|^k f_v^k \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\frac{Q_{v-1}^{k}p_{v}^{k}}{q_{v}^{2k-1}}|t_{v}|^{k}|\lambda_{v}|^{k} f_{v}^{k}\sum_{n=v+1}^{m+1}\frac{n^{k-1}q_{n}^{k}}{Q_{n}^{k}Q_{n-1}}$$

$$= O(1)\sum_{v=1}^{m} v^{k-1}|t_{v}|^{k}\frac{p_{v}^{k}}{q_{v}^{k}}|\lambda_{v}|^{k} f_{v}^{k}$$

$$= O(1)\sum_{v=1}^{m} v^{k-1}|t_{v}|^{k}|\lambda_{v}|^{k}\left(\frac{vp_{v}}{P_{v}}\right)^{k} = O(1).$$

$$\sum_{n=2}^{m+1} n^{k-1}|T_{n3}|^{k} = \sum_{n=2}^{m+1} n^{k-1}\left|\frac{q_{n}}{Q_{n}Q_{n-1}}\sum_{v=1}^{n-1}\frac{Q_{v-1}}{q_{v}}t_{v}\,\Delta\lambda_{v}\,F_{v+1}\right|^{k}$$

$$\leq\sum_{n=1}^{m+1} n^{k-1}\frac{q_{n}^{k}}{Q_{n}^{k}Q_{n-1}}\sum_{v=1}^{n-1}\frac{Q_{v-1}^{k}}{q_{v}^{2k-1}}|t_{v}|^{k}|\Delta\lambda_{v}|^{k} F_{v}^{k}\left(\sum_{v=1}^{n-1}\frac{q_{v}}{Q_{n-1}}\right)^{k}$$

$$= O(1)\sum_{v=1}^{m}\frac{Q_{v-1}^{k}}{q_{v}^{2k-1}}|t_{v}|^{k}|\Delta\lambda_{v}|^{k} F_{v}^{k}\sum_{n=v+1}^{m+1}\frac{n^{k-1}q_{n}^{k}}{Q_{n}^{k}Q_{n-1}}$$

$$= O(1)\sum_{v=1}^{m} v^{k-1}|t_{v}|^{k}|\Delta\lambda_{v}|^{k} F_{v}\frac{1}{q_{v}^{k}} = O(1).$$

$$\sum_{n=1}^{m} n^{k-1}|T_{n4}|^{k} = \sum_{n=1}^{m} n^{k-1}\left|\frac{p_{n}}{Q_{n-1}}t_{n}\lambda_{n}f_{n}\right|^{k}$$

$$= O(1)\sum_{n=1}^{m} n^{k-1}|t_{n}|^{k}|\lambda_{n}|^{k}\left(\frac{q_{n}}{Q_{n-1}}\right)^{k}\left(\frac{np_{n}}{P_{n}}\right)^{k} \quad ,$$

as $\dfrac{q_{n}}{Q_{n-1}} = \dfrac{q_{n}}{Q_{n}-q_{n}} = \dfrac{1}{\dfrac{Q_{n}}{q_{n}}-1}\to 0,$

$$\sum_{n=1}^{\infty} n^{k-1}|T_{n4}|^{k} = O(1)\sum_{n=1}^{m} n^{k-1}|t_{n}|^{k}|\lambda_{n}|^{k}\left(\frac{np_{n}}{P_{n}}\right)^{k} = O(1)$$

$$\sum_{n=2}^{m+1} n^{k-1}|T_{n5}|^{k} = \sum_{n=2}^{m+1} n^{k-1}\left|\frac{p_{n}q_{n}}{P_{n}Q_{n-1}}\sum_{v=1}^{n-1}t_{v}\lambda_{v}\right|^{k}$$

$$\leq\sum_{n=1}^{m+1} n^{k-1}\frac{p_{n}^{k}q_{n}^{k}}{P_{n}^{k}Q_{n-1}}\sum_{v=1}^{n-1}|t_{v}|^{k}|\lambda_{v}|^{k}\frac{1}{q_{v}^{k-1}}\left(\sum_{v=1}^{n-1}\frac{q_{v}}{Q_{n-1}}\right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m}|t_{v}|^{k}|\lambda_{v}|^{k}\frac{1}{q_{v}^{k-1}}\sum_{n=v+1}^{m+1}\frac{n^{k-1}p_{n}^{k}q_{n}^{k}}{P_{n}^{k}Q_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}|t_{v}|^{k}|\lambda_{v}|^{k}\frac{1}{q_{v}^{k-1}}\sum_{n=v+1}^{m+1}\frac{n^{k-1}q_{n}^{k}}{Q_{n}^{k}Q_{n-1}}$$

$$= O(1)\sum_{v=1}^{m} v^{k-1}|t_{v}|^{k}|\lambda_{v}|^{k}\frac{1}{Q_{v}^{k}} = O(1).$$

$$\sum_{n=2}^{m+1} n^{k-1}|T_{n6}|^k = \sum_{n=2}^{m+1} n^{k-1}\left|\frac{p_n q_n}{P_n Q_{n-1}}\sum_{v=1}^{n-1}\frac{Q_{v-1}}{q_v}t_v\,\Delta\lambda_v\right|^k$$

$$\le \sum_{n=2}^{m+1} n^{k-1}\frac{p_n^k q_n^k}{P_n^k Q_{n-1}}\sum_{v=1}^{n-1}\frac{Q_{v-1}^k}{q^{2k-1}}|t_v|^k|\Delta\lambda_v|^k\left(\sum_{v=1}^{n-1}\frac{q_v}{Q_{n-1}}\right)^{k-1}$$

$$= \mathrm{O}(1)\sum_{v=1}^{m}\frac{Q_{v-1}^k}{q_v^{2k-1}}|t_v|^k|\Delta\lambda_v|^k\sum_{n=v+1}^{m+1} n^{k-1}\frac{p_n^k q_n^k}{P_n^k Q_{n-1}}$$

$$= \mathrm{O}(1)\sum_{v=1}^{m} v^{k-1}|t_v|^k|\Delta\lambda_v|^k\frac{1}{q_v^k} = \mathrm{O}(1).$$

Finally

$$\sum_{n=1}^{m} n^{k-1}|T_{n7}|^k = \sum_{n=1}^{m} n^{k-1}\left|\frac{p_n Q_n}{P_n Q_{n-1}}t_n\lambda_n\right|^k$$

$$= \mathrm{O}(1)\sum_{n=1}^{m} n^{k-1}|t_n|^k|\lambda_n|^k\left(\frac{p_n}{P_n}\right)^k = \mathrm{O}(1).$$

This completes the proof of the theorem.

Theorem: 2.2 Let (7) be satisfied and

(16) $$P_v = \mathrm{O}(p_v Q_v),$$

(17) $$Q_n = \mathrm{O}(nq_n).$$

Then necessary conditions for the implication (8) to be satisfied are

$$|\lambda_v| = \mathrm{O}\left(\frac{Q_{v-1}}{1+F_v}\right),\ |\lambda_v| = \mathrm{O}\left(\frac{v^{1-1/k}q_v}{p_v f_v}\right),\ |\Delta\lambda_v| = \mathrm{O}\left(\frac{v^{1-1/k}q_v}{1+F_{v+1}}\right).$$

Proof: For $k \ge 1$ define

$$A^* = \{(a_j): \textstyle\sum a_j \ is\ summable\ |R, q_n|_k\},$$
$$B^* = \{(b_j): \textstyle\sum b_j\lambda_j \ is\ summable\ |(R, q_n)(R, p_n)|_k\}.$$

From (15), we have

(18) $$T_n = \sum_{v=1}^{n}\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)a_v\lambda_v$$

With t_n *and* T_n as defined by (14) and (18), the spaces A^* *and* B^* are BK-spaces with norms defined by

(19) $$\|c\|_1 = \left\{|t_0|^k + \sum_{n=1}^{\infty} n^{k-1}|t_n|^k\right\}^{1/k},$$

(20) $$\|c\|_2 = \left\{|T_0|^k + \sum_{n=1}^{\infty} n^{k-1}|T_n|^k\right\}^{1/k}.$$

respectively. By the hypothesis of the theorem,

(21) $$\|c\|_1 < \infty \ \Rightarrow\ \|c\|_2 < \infty.$$

The inclusion map $i: A^* \to B^*$ defined by i(a) = a is continuous since A^* and B^* are BK-spaces. By the closed graph theorem, there exist a constant K>0 such that

(22) $$\|c\|_2 \le K\,\|c\|_1 .$$

Let e_n denote the nth coordinate vector. From (14) and (18), with (a_n) defined by $a_n = e_n - e_{n+1}$, $n = v$, $a_n = 0$ otherwise, we have

$$t_n = \begin{cases} 0, & n < v \\ \dfrac{q_v}{Q_v}, & n = v \\ -\dfrac{q_n q_v}{Q_n Q_{n-1}}, & n > v\ . \end{cases}$$

and

$$T_n = \begin{cases} 0, & n < v \\ \left(\dfrac{q_v F_v}{Q_v Q_{v-1}} + \dfrac{p_v q_v}{P_v Q_{v-1}}\right)\lambda_v & n = v \\ \Delta_v\left(\left(\dfrac{q_n F_v}{Q_n Q_{n-1}} + \dfrac{p_n q_n}{P_n Q_{n-1}}\right)\lambda_v\right), & n > v. \end{cases}$$

From (19) and (20), we have

(23) $$\|c\|_1 = \left\{ v^{k-1}\left(\frac{q_v}{Q_v}\right)^k + \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n q_v}{Q_n Q_{n-1}}\right)^k \right\}^{1/k},$$

(24) $$\|c\|_2 = \left\{ v^{k-1}\left|\left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}}\right)\lambda_v\right|^k + \sum_{n=v+1}^{\infty} n^{k-1}\left|\Delta_v\left(\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)\lambda_v\right)\right|^k \right\}^{1/k}$$

Applying (22), we obtain

(25) $$v^{k-1}\left|\left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}}\right)\lambda_v\right|^k + \sum_{n=v+1}^{\infty} n^{k-1}\left|\Delta_v\left(\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)\lambda_v\right)\right|^k$$
$$= O(1)\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k + \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n q_v}{Q_n Q_{n-1}}\right)^k \right).$$

As the R.H.S of (25), by (7), is

$$= O(1)\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k + \frac{q_v^k}{Q_v^{k-1}} \sum_{n=v+1}^{\infty} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} \right)$$

$$= O(1)\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k + \left(\frac{q_v}{Q_v}\right)^{k-1} v^{k-1}\left(\frac{q_v}{Q_v}\right)^k \right)$$

$$= O\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k \right),$$

and the fact that each term of the L.H.S of (25) is $O\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k \right)$, we obtain

$$v^{k-1}\left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}}\right)^k |\lambda_v|^k = O\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k \right),$$

which implies by (16)

$$\left(\frac{q_v}{Q_v Q_{v-1}}\right)^k (1+F_v)^k |\lambda_v|^k = O\left(\frac{q_v}{Q_v}\right)^k,$$

that is

$$|\lambda_v| = O\left(\frac{Q_{v-1}}{1+F_v}\right).$$

Also, we have, by (25),

$$(26) \quad \sum_{n=v+1}^{\infty} n^{k-1}\left|\left(\frac{q_n p_v f_v}{Q_n Q_{n-1}}\right)\lambda_v + \left(\frac{q_n F_{v+1}}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)\Delta\lambda_v\right|^k = O\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k \right).$$

The above, via the linear independence of λ_v *and* $\Delta\lambda_v$, implies

$$\sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n F_{v+1}}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}}\right)^k |\Delta\lambda_v|^k = O\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k \right)$$

$$|\Delta\lambda_v|^k (1+F_{v+1})^k \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n}{Q_n Q_{n-1}}\right)^k = O\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k \right). \qquad \text{(by (16))}$$

As by (17), via the mean value theorem,

$$\frac{1}{Q_v^k} = \sum_{n=v+1}^{\infty} \Delta\left(\frac{1}{Q_{n-1}^k}\right) = O(1) \sum_{n=v+1}^{\infty} \frac{|\Delta Q_{n-1}^k|}{Q_n^k Q_{n-1}^k} = O(1) \sum_{n=v+1}^{\infty} \frac{Q_{n-1}^{k-1} q_n}{Q_n^k Q_{n-1}^k} = O(1) \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n}{Q_n Q_{n-1}}\right)^k, \text{ then,}$$

$$|\Delta\lambda_v|^k (1+F_{v+1})^k \frac{1}{Q_v^k} = O\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k \right),$$

which implies

$$\Delta\lambda_v = O\left(\frac{v^{1-1/k} q_v}{1+F_{v+1}}\right).$$

Also, by (26),

$$\sum_{n=v+1}^{\infty} n^{k-1}\left|\frac{q_n p_v f_v}{Q_n Q_{n-1}}\lambda_v\right|^k = O\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k\right),$$

$$p_v^k f_v^k \left|\lambda_v\right|^k \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{q_n}{Q_n Q_{n-1}}\right)^k = O\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k\right),$$

$$p_v^k f_v^k \left|\lambda_v\right|^k \frac{1}{Q_v^k} = O\left(v^{k-1}\left(\frac{q_v}{Q_v}\right)^k\right),$$

which implies

$$\lambda_v = O\left(\frac{v^{1-1/k} q_v}{p_v f_v}\right).$$

3. APPLICATIONS:

Corollary: 3.1 Let $k \geq 1$. Define

(27) $$f_v = \sum_{r=v}^{n} \frac{q_r}{r}, \; F_v = \sum_{r=v}^{n} f_r .$$

Let

(28) $$Q_v = O(v),$$

(29) $$f_v = O(q_v) .$$

Then sufficient conditions for the implication

(30) $\sum a_n$ is summable $\left|R, q_n\right|_k \Rightarrow \sum a_n \lambda_n$ is summable $\left|(R, q_n)(C,1)\right|_k$ are

(31) $$\left|\lambda_n\right| < Q_n ,$$

(32) $$\left|\lambda_v\right| f_v = O(1),$$

(33) $$\left|\lambda_v\right| F_v = O(Q_v) ,$$

(34) $$\left|\Delta\lambda_v\right| F_v = O(q_v),$$

(35) $$\left|\Delta\lambda_v\right| = O(q_v).$$

Proof: Follows from theorem 1 by putting $p_n = 1$ for all n.

Corollary: 3.2 Let $k \geq 1$. Define

(36) $$f_v = \sum_{r=v}^{n} \frac{1}{P_r}, \; F_v = \sum_{r=v}^{n} p_r f_r .$$

Let

(37) $$v p_v = O(P_v),$$

(38) $$P_v f_v = O(v).$$

Then sufficient conditions for the implication

(39) $\sum a_n$ is summable $\left|C,1\right|_k \Rightarrow \sum a_n \lambda_n$ is summable $\left|(C,1)(R, p_n)\right|_k$

are

(40) $$\left|\lambda_n\right| < n ,$$

(41) $\quad v p_v |\lambda_v| = O(P_v),$

(42) $\quad |\lambda_v| F_v = O(v),$

(43) $\quad |\Delta\lambda_v| F_v = O(1),$

(44) $\quad |\Delta\lambda_v| = O(1).$

Proof: This follows from theorem 1, by putting $q_n = 1$ for all n, noticing that (7) for $q_n = 1$ is obviously satisfied as

$$\sum_{n=v+1}^{\infty} \frac{1}{n(n-1)} = \sum_{n=v+1}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = \frac{1}{v}.$$

Corollary: 3.3 Let f_v, F_v be as defined in (27). Let (7) and (17) be satisfied and

(45) $\quad v = O(Q_v).$

Then necessary conditions for the implication (30) are

$$\lambda_v = O\left(\frac{Q_{v-1}}{1+F_v} \right), \lambda_v = O\left(\frac{v^{1-1/k} q_v}{f_v} \right) \ \Delta\lambda_v = O\left(\frac{v^{1-1/k} q_v}{1+F_{v+1}} \right)$$

Proof: Follows from theorem 4 by putting $p_n = 1$ for all n.

Corollary: 3.4 Let f_v, F_v be as defined in (36). Let

(46) $\quad P_v = O(v p_v).$

Then necessary conditions for the implication (39) are

$$\lambda_v = O\left(\frac{v}{1+F_v} \right), \lambda_v = O\left(\frac{v^{1-1/k}}{p_v f_v} \right), \Delta\lambda_v = O\left(\frac{v^{1-1/k}}{1+F_{v+1}} \right).$$

Proof: Follows from theorem 4 by putting $q_n = 1$ for all n.

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