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APPROXIMATE CYCLIC MODULE AMENABILITY OF BANACH ALGEBRAS

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ABSTARCT

In this paper, we first introduce the concept of approximately cyclic module amenability for Banach algebras and then study the hereditary properties of approximately cyclic module amenability of Banach algebras. Also, the relationship between approximately cyclic $\mathfrak{A}$-module amenability of I, A/I and A, where A is Banach algebra and I is closed ideal and $\mathfrak{A}$-submodule of A has been studied.

Keywords:*Approximately inner derivation, Cyclic derivation, Module amenability, Approximately module amenability, Cyclic module amenability.*

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1.INTRODUCTION

Let A be a Banach algebra, and X be a Banach A-bimodule. A bounded linear map $D: A \longrightarrow X$ is called a derivation if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in A).$$

For each $x \in X$, $ad_x: A \longrightarrow X$ defined with

$$ad_x(a) = a \cdot x - x \cdot a \qquad (a \in A),$$

is derivation. Derivations of this form are called inner derivations. If there exists a net $(x_\nu) \subseteq X$ such that

$$D(a) = \lim_\nu (a \cdot x_\nu - x_\nu \cdot a) \qquad (a \in A),$$

then D is called approximately inner derivation.

A Banach algebra A is called amenable if for any Banach A-bimodule X, every derivation $D: A \longrightarrow X^*$ is inner, and it is approximately amenable if every derivation $D: A \longrightarrow X^*$ is approximately inner, where X^* is the dual of X. Also, A is called weak amenable if every derivation $D: A \longrightarrow A^*$ is inner and it is approximately weak amenable if every derivation $D: A \longrightarrow A^*$ is approximately inner.

The concepts of amenability and weak amenability of Banach algebra was defined and studied by Johnson in [6] and Bade, Curtis and Dales in [3], respectively. Also the concepts of approximately amenability and approximately weak amenability of Banach algebra was Introduced by Ghahramani and Loy in [4]. It is clear that, if A is approximately amenable, then approximatly weak amenable but the vice versa is not hold (see [4, Theorem 3.2]). Obviously, if A is weakly amenable then it is approximately weakly amenable. However, the converse is not true in general, as it was shown in [4, Example 6.2] . Note that, if A is commutative A is weakly amenable if and only if it approximately weakly amenable (because in this case the only inner derivation from A to A^* is zero).

A derivation $D: A \longrightarrow A^*$ is called cyclic if

$$[D(a)](b) + [D(b)](a) = 0 \qquad (a, b \in A).$$

A Banach algebra A is called cyclic amenable if every cyclic derivation $D: A \longrightarrow A^*$ is inner. Also, A is called approximately cyclic amenable if every cyclic derivation $D: A \longrightarrow A^*$ is approximatly inner.

The concept of cyclic amenability was presented by Grønbæk in [5]. After that, others authors rarely investigated the cyclic amenability and the approximately cyclic amenability of Banach algebras. For example, the second author has studied the cyclic amenability and approximately cyclic amenability of triangular Banach algebras in [8]. Also Shojaee and Bodaghi generalized it in [9] to follow the results of Ghahramani and Loy [4].

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On the other hand, the concepts of module amenability and weak module amenability for Banach algebras which is Banach module over another Banach algebra, has been developed by Amini in [1] and Amini along with Bagha in [2], respectively.

In this paper, we define the concept of approximately cyclic module amenability for Banach algebras which is weaker than the concepts approximately cyclic amenability and approximately weak module amenability. Indeed, we indicate there exist approximately cyclic module amenable Banach algebras which are not approximately cyclic amenable and approximately weak module amenable. Then we investigate relationship between approximately cyclic $\mathfrak{A}$-module amenability of I, A/I and A, where I is closed ideal and $\mathfrak{A}$-submodule of A.

2. PRELIMINARIES

Let $\mathfrak{A}$ and A be Banach algebras such that A is a Banach $\mathfrak{A}$-bimodule with compatible actions, and let X be a left Banach A-module and a Banach $\mathfrak{A}$-bimodule with the following compatible actions is called left Banach A-$\mathfrak{A}$-module:

$$\alpha\cdot(a\cdot x)=(\alpha\cdot a)\cdot x,\ \ a\cdot(\alpha\cdot x)=(a\cdot\alpha)\cdot x,\ \ a\cdot(x\cdot\alpha)=(a\cdot x)\cdot\alpha,$$

for all $a\in A,\alpha\in\mathfrak{A}$ and $x\in X$. The right Banach A-$\mathfrak{A}$-module is defined semilarly. If X be two-side Banach A-$\mathfrak{A}$-module, it is called Banach A-$\mathfrak{A}$-module. Also, X is called a commutative (bi-commutative) Banach A-$\mathfrak{A}$-module, if $\alpha\cdot x=x\cdot\alpha(a.x=x.a)$ for all $\alpha\in\mathfrak{A},a\in A$ and $x\in X$. If X is a (commutative) Banach A-$\mathfrak{A}$-module, then so is X^*, where the actions of A and $\mathfrak{A}$ on X^* are defined as usual:

$$\langle f\cdot\alpha,x\rangle=\langle f,\alpha\cdot x\rangle,\quad \langle f\cdot a,x\rangle=\langle f,a\cdot x\rangle,$$
$$\langle\alpha\cdot f,x\rangle=\langle f,x\cdot\alpha\rangle,\quad \langle a\cdot f,x\rangle=\langle f,x\cdot a\rangle,$$
$$(a\in A,\alpha\in\mathfrak{A},\ x\in X,f\in X^*).$$

Notice that, if A is a commutative Banach $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is a commutative Banach A-$\mathfrak{A}$-module. In this case, the dual space A^* is also a commutative Banach A-$\mathfrak{A}$-module.

A bounded map $D: A\longrightarrow X$ is called a $\mathfrak{A}$-module derivation if for all $a,b\in A$ and $\alpha\in\mathfrak{A}$:

$$D(a\pm b)=D(a)\pm D(b), D(\alpha\cdot a)=\alpha\cdot D(a), D(a\cdot\alpha)=D(a)\cdot\alpha,$$

and

$$D(ab)=a\cdot D(b)+D(a)\cdot b.$$

Moreover, $\mathfrak{A}$-module derivation $D: A\longrightarrow A^*$ is called cyclic if it satisfy the following condition $[D(a)](b)+[D(b)](a)=0$.

Definition 1: A Bananch algebra A is called $\mathfrak{A}$-module amenable if for any commutative Banach A-$\mathfrak{A}$-module X, each $\mathfrak{A}$-module derivation $D: A\longrightarrow X^*$ is inner, and it is approximatly $\mathfrak{A}$-module amenable if every $\mathfrak{A}$-module derivation $D: A\longrightarrow X^*$ is approximately inner.

A commutative Bananch $\mathfrak{A}$-bimodule A is called weak $\mathfrak{A}$-module amenable, if every $\mathfrak{A}$-module derivation $D: A\longrightarrow A^*$ is inner, and it is approximatly weak $\mathfrak{A}$-module amenable if every $\mathfrak{A}$-module derivation $D: A\longrightarrow A^*$ is approximately inner. The interested reader can be founded the difference between the above definitions in Example 2.4 of [7]. Also, in Definition 2.1 of [7] we define cyclic $\mathfrak{A}$-module amenable as followes. A commutative Bananch $\mathfrak{A}$-module A, is called cyclic $\mathfrak{A}$-module amenable if every cyclic $\mathfrak{A}$-module derivation $D: A\longrightarrow A^*$ is inner.

Definition 2: Let A be Bananch algebra, which is a $\mathfrak{A}$-bimodule, if each cyclic $\mathfrak{A}$-module derivation $D: A\longrightarrow A^*$ is approximately inner, it is called approximately cyclic $\mathfrak{A}$-module amenable.

Remark 3: Let A be Banach algebra, the approximately weak amenability implies that the approximately cyclic amenability which is conclude the approximately cyclic module amenability of A. Also, the approximately weak module amenability deduce the approximately cyclic module amenability of A.

At the Example 2.4 of [7] the difference between the concepts of cyclic module amenability and cyclic amenability is shown. Also, the authors presented the Banach algebras that are cyclic module amenable but are not weak module amenable.

In the following example, as similar as Example 6.2 of [4], we show that the concept of approximately cyclic module amenability is not equal to the concept of cyclic module amenability of A (as an $\mathfrak{A}$-module when $\mathfrak{A}=\mathbb{C}$).

Example 4: Let M_n be the set of all square matrices with the following norm; $\parallel (a_{ij}) \parallel_2 = (\sum_{i,j} |a_{ij}|^2))^{\frac{1}{2}}$, which is algebric norm. Now, we define

$$\phi: M_n \longrightarrow M_n^*$$

where $A: M_n \longrightarrow \mathbb{C}$ to be considered with $A(B) = \sum_{i,j} a_{ij} b_{ij}$. We know that, ϕ is an isometrical homomorphism. On the other hand, for every $A, B \in M_n$ and $E \in M_n^*$, we can show that

$$\langle A \cdot E, B \rangle = \langle E \cdot A^T, B \rangle \qquad \text{and} \qquad \langle E \cdot A, B \rangle = \langle A \cdot E^T, B \rangle.$$

Therefore, $ad_P(A) = A \cdot P - P \cdot A = P \cdot A^T - A^T \cdot P$.

In paticular, $n = 2$, take

$$P_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_n^* \qquad \text{and} \qquad \parallel P_1 \parallel_2 = 2.$$

Moreover,

$$ad_{P_1}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -b-c & -a+d \\ a-d & c+d \end{bmatrix}.$$

As the same way, for any $n \in \mathbb{N}$, we have

$$P_{n+1} := \begin{bmatrix} 0 & -P_n \\ P_n & 0 \end{bmatrix} \in M_{2^n} \qquad \text{and} \qquad \parallel ad_{P_n} \parallel = 2^{\frac{n}{2}} \leq 2.$$

Now, put $A_n = M_{2^n}^{\sharp}$ and define:

$$D: C_0(A_n) \longrightarrow Ml^1(A_n^*) = C_0(A_n)^*$$
$$A \mapsto \frac{ad_{P_n}(x_n)}{n^2}$$

Similar to Example 6.2 of [4], D is cyclic derivation which is not inner.

3. MAIN RESULTS

Let A be a Banach algebra and commutative Banach $\mathfrak{A}$-module with compatible actions, and let I be a closed ideal of A. In general, I and A/I are not necessarily Banach $\mathfrak{A}$-module with compatible actions. Throughout this section we assume that I is closed ideal and Banach $\mathfrak{A}$-submodule of A. In this case both I and A/I are commutative Banach $\mathfrak{A}$-module with the canonical actions.

In this section, we study relationship between approximately cyclic $\mathfrak{A}$-module amenability of I, A/I and A, where I is closed ideal and $\mathfrak{A}$-submodule of A.

Definition 5: Let I be a closed ideal in A. We say that I has the trace extension property if for each $\lambda \in I^*$ with $a.\lambda = \lambda.a \ (a \in A)$ there is $\Lambda \in A^*$ such that $\Lambda|_I = \lambda$ and $a \cdot \Lambda = \Lambda \cdot a$ for every $a \in A$. Also I has the approximately trace extension property if for each $\lambda \in I^*$ with $a.\lambda = \lambda.a \ (a \in A)$ there is a net $(\Lambda_\alpha) \subseteq A^*$ such that for each $\alpha, \Lambda_\alpha|_I = \lambda$ and

$$a \cdot \Lambda_\alpha - \Lambda_\alpha \cdot a \longrightarrow 0 \qquad (a \in A).$$

Theorem 6: Let $\frac{A}{I}$ be approximately cyclic $\mathfrak{A}$-module amenable. Then I has the approximately trace extension property.

Proof: Let $\lambda \in I^*$, such that $a \cdot \lambda = \lambda \cdot a$ for every $a \in A$. Take $\theta \in A^*$ with $\theta|_I = \lambda$. Define

$$D: \frac{A}{I} \longrightarrow \left(\frac{A}{I}\right)^*$$
$$\bar{a} \mapsto a \cdot \theta - \theta \cdot a.$$

As similar as proof of Proposition 9 of [7], we can show that D is cyclic module derivation. Since, $\frac{A}{I}$ is approximately cyclic $\mathfrak{A}$-module amenable, there exists a net $(\lambda_\alpha) \in \left(\frac{A}{I}\right)^* = I^\perp$ such that

$$D(\bar{a}) = \lim_\alpha a \cdot \lambda_\alpha - \lambda_\alpha \cdot a \qquad (a \in A).$$

Now, let $\Lambda_\alpha = \theta - \lambda_\alpha \in A^*$. For each α we have,

$$\Lambda_\alpha|_I = (\theta - \lambda_\alpha)|_I = \theta|_I - \lambda_\alpha|_I = \theta|_I - 0 = \lambda,$$

and for every $a \in A$,

$$\begin{aligned}\lim_\alpha a \cdot \Lambda_\alpha - \Lambda_\alpha \cdot a &= \lim_\alpha (\theta - \lambda_\alpha) \cdot a - a \cdot (\theta - \lambda_\alpha)\\ &= \lim_\alpha (\theta \cdot a - a \cdot \theta) + (a \cdot \lambda_\alpha - \lambda_\alpha \cdot a)\\ &= \lim_\alpha (\theta \cdot a - a \cdot \theta) + (\overline{a} \cdot \lambda_\alpha - \lambda_\alpha \cdot \overline{a})\\ &= -D(\overline{a}) + ad_{\lambda_\alpha}(\overline{a})\\ &= -D(\overline{a}) + D(\overline{a}) = 0.\end{aligned}$$

Therefore, I has the approximately trace extension property.

Theorem 7: Let A be cyclic $\mathfrak{A}$-module amenable and I has the approximately trace extension property. Then $\frac{A}{I}$ is approximately cyclic $\mathfrak{A}$-module amenable.

Proof: Let $D: \frac{A}{I} \longrightarrow (\frac{A}{I})^*$ be cyclic $\mathfrak{A}$-module derivation and $\pi: A \longrightarrow \frac{A}{I}$ be the quotient map. Take $\widetilde{D} = \pi^* \circ D \circ \pi: A \to A^*$. It is clear that $\widetilde{D}$ is cyclic $\mathfrak{A}$-module derivation (see Proposition 10 of [7]). But since A is cyclic $\mathfrak{A}$-module amenable, there exists $\lambda \in A^*$ with

$$\widetilde{D}(a) = ad_\lambda(a) = a \cdot \lambda - \lambda \cdot a \qquad (a \in A).$$

Clearly $\widetilde{D}(a)|_I = 0$. Set $\hat{\lambda} = \lambda|_I$. For every $a \in A$, we have

$$\begin{aligned}a \cdot \hat{\lambda} - \hat{\lambda} \cdot a &= (a \cdot \lambda - \lambda \cdot a)|_I\\ &= \widetilde{D}(a)|_I\\ &= 0.\end{aligned}$$

Since I has the approximately trace extension property, there exists a net $(\Lambda_\alpha) \subseteq A^*$ such that for any α; $\Lambda_\alpha|_I = \hat{\lambda}$ and

$$\lim_\alpha a \cdot \Lambda_\alpha - \Lambda_\alpha \cdot a = 0 \qquad (a \in A).$$

Now $\lambda - \Lambda_\alpha \in I^\perp$ for each α and also,

$$D(\overline{a}) = \lim_\alpha a \cdot (\lambda - \Lambda_\alpha) - (\lambda - \Lambda_\alpha) \cdot a.$$

For see this let $a, b \in A$,

$$\begin{aligned}\langle \overline{b}, D(\overline{a})\rangle &= \langle b, (\pi^* \circ D \circ \pi)(a)\rangle\\ &= \langle b, \widetilde{D}(a)\rangle\\ &= \langle b, a \cdot \lambda - \lambda \cdot a\rangle\\ &= \lim_\alpha \langle b, a \cdot (\lambda - \Lambda_\alpha) - (\lambda - \Lambda_\alpha) \cdot a\rangle\\ &= \lim_\alpha \langle \overline{b}, \overline{a} \cdot (\lambda - \Lambda_\alpha) - (\lambda - \Lambda_\alpha) \cdot \overline{a}\rangle.\end{aligned}$$

Thus D is approximately inner. This shows that, $\frac{A}{I}$ is approximately cyclic $\mathfrak{A}$-module amenable.

Theorem 8: Let $\frac{A}{I}$ be approximately cyclic $\mathfrak{A}$-module amenable and I be cyclic $\mathfrak{A}$-module amenable and $\overline{I^2} = I$. Then A is approximately cyclic $\mathfrak{A}$-module amenable.

Proof: Let $D: A \longrightarrow A^*$ be cyclic $\mathfrak{A}$-module derivation, define $\widehat{D} := \boldsymbol{\iota}^* \circ D \circ \boldsymbol{\iota}$, where, $\boldsymbol{\iota}: I \hookrightarrow A$ is the natural embedding. We show that $\widehat{D}$ is $\mathfrak{A}$-module derivation. Suppose $\alpha \in \mathfrak{A}$ and $a, b \in A$,

$$\begin{aligned}\langle b, \widehat{D}(\alpha \cdot a)\rangle &= \langle b, (\boldsymbol{\iota}^* \circ D \circ \boldsymbol{\iota})(\alpha \cdot a)\rangle = \langle b, D(\alpha \cdot a)\rangle\\ &= \langle b, \alpha \cdot D(a)\rangle = \langle b \cdot \alpha, D(a)\rangle\\ &= \langle b \cdot \alpha, (\boldsymbol{\iota}^* \circ D \circ \boldsymbol{\iota})(a)\rangle = \langle b \cdot \alpha, \widehat{D}(a)\rangle\\ &= \langle b, \alpha \cdot \widehat{D}(a)\rangle.\end{aligned}$$

So $\widehat{D}: I \longrightarrow I^*$ cyclic $\mathfrak{A}$-module derivation and since I is cyclic $\mathfrak{A}$-module amenable, there exists $\lambda \in I^*$ with

$$\widehat{D}(i) = (\boldsymbol{\iota}^* \circ D)(i) = ad_\lambda(i) \qquad (i \in I).$$

By using the Hahn-Banach extension Theorem, if $\tilde{\lambda} \in A^*$ to be the extension of λ, we can define $\widetilde{D} = D - ad_{\tilde{\lambda}}$. According to our findings in [7, Proposition 13], $\operatorname{Im} \widetilde{D} \subseteq I^\perp$ and the map

$$\begin{aligned}&\mathcal{D}: \frac{A}{I} \longrightarrow (\frac{A}{I})^* = I^\perp\\ &\overline{a} \longmapsto \widetilde{D}(a)\end{aligned}$$

is cyclic $\mathfrak{A}$-module derivation and since $\frac{A}{I}$ is approximately cyclic $\mathfrak{A}$-module amenable, there exists a net $(\lambda_\alpha) \subseteq (\frac{A}{I})^*$ such that

$$\mathcal{D}(\overline{a}) = \lim_\alpha \overline{a} \cdot \lambda_\alpha - \lambda_\alpha \cdot \overline{a} \qquad (a \in A).$$

Thus

$$\widetilde{D}(a) = \lim_{\alpha} \bar{a} \cdot \lambda_\alpha - \lambda_\alpha \cdot \bar{a} \qquad (a \in A).$$

With some simple calculations, it can be shown that

$$D(a) = \lim_{\alpha} a \cdot \omega_\alpha - \omega_\alpha \cdot a \qquad (a \in A),$$

where $\omega_\alpha := \lambda_\alpha + \lambda$. Hence, D is approximately inner. It now follows that A is approximately cyclic $\mathfrak{A}$-module amenable.

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