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POLARITY IN SIGNED ORDER – PRESERVING AND ORDER – DECREASING SEMIGROUP

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ABSTRACT

Let α be a transformation from the set$X_n \longrightarrow X_n^*$, then the signed (partial) transformation semigroup is defined in the $\alpha: dom(\alpha) \subseteq X_n \longrightarrow Im(\alpha) \subset X_n^*$ where $X_n = \{1,2,3,\cdots,n\}$ and $X_n^* = \{-n,\cdots,-3,-2.-1.0,1,2,3,\cdots,n\}$. The paper aimed at investigate the polarity of elements in these semigroup.

Keywords: *polarity, semigroup, signed order – preserving semigroup, signed order decreasing – semigroup.*

INTRODUCTION AND PRELIMINARY

[4] studied the semigroups of order – preserving and order – preserving of a finite set $X_n = \{1,2,3,\cdots\}$. A map $\alpha: X \longrightarrow X_n^*$ is called order – decreasing, D_n of all i in $X, i\alpha \leq i$. The semigroups of all order – decreasing maps is of cardinality $n!$. A general study of D_n was initiated by [17]. A mapping is called order – preserving if for all i, j in $\{1,2,3,\cdots\}$, $i \leq j \Longrightarrow i\alpha \leq \alpha j$ where $i\alpha, \alpha j \in dom(\alpha)$. The semigroup of order – preserving full transformation of X_n will be denoted by O_n. [4] showed that the order of $|\ O_n| = \binom{2n-1}{n-1}$

[7] obtains some results concerning the semigroup of all maps that are both order – preserving and order – decreasing and showed that $|D_n \cap O_n| = |C_n|$ the Catalan numbers

Let ST_n be signed full transformation semigroup on $\alpha: X_n \longrightarrow X_n^*$ under the usual composition. The signed (partial) transformation semigroups defined in the form$\alpha: dom(\alpha) \subseteq Im(\alpha) \subset X_n^*$. The domain may be empty. We call α signed transformation order – decreasing SD_n if $|i\alpha| \leq i$ for all i in $dom(\alpha)$ and α is signed order – preserving SO_n if $i \leq j \Longrightarrow |i\alpha| \leq |j\alpha|$ for all $i, j \in dom(\alpha)$. The semigroup of all maps that are both signed order – preserving and signed order – decreasing are represents by SC_n and $SC_n = SD_n \cap SO_n. Dom(\alpha)$ stands for the domain of α while the $Im(\alpha)$ as image of α as defined by [5].

[15] initiated the study of signed symmetric group while. [11] studied the signed semigroup of full, partial and partial one – one transformation semigroups. The general studied of SD_n, SO_n and SC_n was initiated by [10], [11] , [12], [13], [14]. He studied the order, number of idempotent, nilpotent, self - inverse, decomposition of SD_n, SO_n and SC_n respectively.

The following known results and theorems are very useful to this work.

Theorem 2.1[11] Theorem 4.1.1]. Let $S = SO_n$, then for $n \geq 1$. $|\ S\ | = 2^n \binom{2n-1}{n-1}$

Theorem 2.2[11] Theorem 4.1.2]. Let $S = SPO_n$, then $|\ S\ | = \sum_{k=0}^{n} \binom{n}{k}^3 2^k$

Theorem 2.3[11] Theorem 4.1.3]. Let $S = SIO_n$, then $|\ S\ | = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}$

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Theorem 2.4[11] Theorem 4.4.1]. Let $S = SDO_n$, then $|\ S\ | = n!\sum_{k=0}^{n}\binom{n}{k}$

Theorem 2.5[11] Theorem 4.4.2]. Let $S = ID_n$, then $|\ S\ | = (k+1)!\binom{n}{k}$

Theorem 2.[11]Theorem 4.8.1]. Let $S = C_n$, then $|\ S\ | = \frac{1}{n}\binom{2n}{n-1} = C_n$

Theorem 2.7[11] Theorem 4.8.2]. Let $S = SC_n$, then $|\ S\ | = \frac{1}{n}\binom{2n}{n-1}\sum_{k=0}^{n}\binom{n}{k}$

Theorem 2.8[11] Theorem 4.8.3]. Let $S = SPC_n$, then $|\ S\ | = \sum_{k=0}^{n}\binom{n}{k}^3\binom{2n}{k}$

Theorem 2.9[11] Theorem 4.8.4]. Let $S = SIC_n$, then $|\ S\ | = \sum_{k=0}^{n}\binom{n}{k}\binom{2^k}{k}$

METHODOLOGY

Let PSO_n, PSD_n, PSC_n be the polarity of signed order – preserving, signed order – decreasing and both signed order – preserving and signed order – decreasing transformation semigroup respectively define on $\alpha: X_n \longrightarrow X_n^*$

Polarity of element in signed order – preserving semigroup

Elements in PSO_1 is

$$|\ PSO_1\ | = \left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} = 1$$

Elements in PSO_2

$$|\ PSO_2\ | = \left\{\begin{matrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \\ \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix} \end{matrix}\right\} = 9$$

$$|\ Im(\alpha^-)\ | = \left\{\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}\right\} = 3$$

$$|\ Im(\alpha^*)\ | = \left\{\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 2 \end{pmatrix}\right\} = 6$$

Polarity of element in signed order – decreasing semigroup

Elements in PSD_1 is

$$|\ PSD_1\ | = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Elements in PSD_2

$$|\ PSD_2\ | = \left\{\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\right\}$$

$$|\ Im(\alpha^-)\ | = \left\{\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\right\} = 2$$

$$|\ Im(\alpha^*)\ | = \left\{\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}\right\} = 4$$

Polarity of element in both signed order – preserving and order decreasing semigroup

Elements in PSC_1 is

$$|\ PSC_1\ | = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Elements in PSC_2

$$|\ PSC_2\ | = \left\{\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\right\}$$

$$|\ Im(\alpha^-)\ | = \left\{\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\right\} = 2$$

$$|\ Im(\alpha^*)\ | = \left\{\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}\right\} = 4$$

The following table displays element of PSO_n

Table-3.1: The value of elements in PSO_n

n	$\mid Im(\alpha^-)\mid$	$\mid Im(\alpha^*)\mid$	$\mid PSO_n\mid = \binom{2n-1}{n-1}(2^n-1)$
1	1	−	1
2	3	6	9
3	10	60	70
4	35	490	525
5	126	3780	3906

$\mid Im(\alpha^-)\mid$ = number of the elements with negative integers only in the image of α
$\mid Im(\alpha^*)\mid$ = number of the elements with positive integers only in the image of α

Theorem 3.1: Let $S = PSO_n$, then $\mid S\mid = \binom{2n-1}{n-1}(2^n-1)$

Proof: Let $\alpha \in S$ and the $lm(\alpha) \subset X_n^*$ and $X_n \subset X_n^*$ where X_n^* is the set of elements with the positive and negative only the image of α. Choices some images i from $X_n^* = \{-n, \cdots, -3, -2. -1.0,1,2,3, \cdots, n\}$ such as that the $Im(\alpha^-) = \{-i, -i\} \in X_n^*$. Since the semigroup is a full transformation the elements of $dom(\alpha)$ can be chosen from X_n^* in $\binom{n}{k}$ which is equivalent to $(2^n - 1)$ elements. If the $\mid Im(\alpha^-)\mid = \binom{2n-1}{n-1}$ which is equivalents to $\mid SO_n\mid$, then follows by applying the product rule . hence the result follows.

Table-3.2: Values of elements in PSD_n

n	$\mid Im(\alpha^-)\mid$	$\mid Im(\alpha^*)\mid$	$\mid PSD_n\mid = n!\,(2^n-1)$
1	1	−	1
2	2	4	6
3	6	36	42
4	24	336	360
5	120	3600	3720
6	720	44640	45360

Theorem 3.2: Let $S = PSD_n$, then $\mid S\mid = n!\,(2^n - 1)$.

Proof: Let $\alpha: X \longrightarrow X_n^*$, then $Im(\alpha) \subset X_n^*$ iff $Im(\alpha^*) \subset X_n^*$, for each $\alpha \in PDO_n$ we have $lm(\alpha^-) = n!$. Since the $Im(\alpha) = \{i, -i\}$ where $i = 1,2,3, \ldots$ If the $lm(\alpha^-) = 1$, then $|\alpha S| = n!$ while $|\alpha S| = 2^n$ for $lm(\alpha^*) = 2$. Hence we have $n!\,(2^n - 1)$ elements

Table-3.3: Values of elements in PSC_n

n	$\mid Im(\alpha^-)\mid$	$\mid Im(\alpha^*)\mid$	$\mid PSC_n\mid = \frac{1}{n}\binom{2n}{n-1}\left[\sum_{k=0}^{n}\binom{n}{k}-1\right]$
1	1	−	1
2	2	4	6
3	5	30	35
4	14	196	210
5	42	1260	1302

Theorem 3.3: Let $S = PSC_n$ and if $\alpha \in PSC_n$ then $\mid S\mid = \frac{1}{n}\binom{2n}{n-1}\left[\sum_{k=0}^{n}\binom{n}{k} - 1\right]$

Proof: It follows from Theorem 2.7. Let $\alpha \in PSC_n$ and $\alpha: X_n \longrightarrow X_n^*$ where $X_n \subset X_n^*$. First observe that $\frac{1}{n}\binom{2n}{n-1} = |C_n|$ where C_n is the nthcatalan number. [6] denoted $|C_n| = |O_n \cap D_n|$ and thus $|PSC_n| = |PSO_n \cap PSDn$. If $doma \subseteq Xn$ and $lma \subset Xn*$ and $lma* \in Xn*$ then $lma- = Cn$ from the table 3.3. Since k elements from the $dom(\alpha)$ in a set can be chosen from X_n in $\binom{n}{k}$ ways and this equivalents to 2^n. If the $Im(\alpha^*) = \{i, -i\}$or $Im(\alpha^*) = \{-i, i\}$ or $Im(\alpha^*) = \{i, -i\}$ then each element from X_n taken could occurs in $2^n - 1$ ways. Hence multiplying and summing over n, gives the results.

SUMMARY OF THE RESULTS

The following results with sequences were obtained for all n.

1. Let $S = PSO_n$, then$|\, S \,| = \binom{2n-1}{n-1}(2^n - 1)$, which generate the sequence $1, 9, 70, 525$,3906 , . . .
2. Let $S = PSD_n$, then $|\, S \,| = n!\,(2^n - 1)$, which generate the sequence1, 6, 42, 360, 3720, 45360, . . .
3. Let $S = PSC_n$, then $|\, S \,| = \frac{1}{n}\binom{2n}{n-1}\left[\sum_{k=0}^{n}\binom{n}{k} - 1\right]$, which generate the sequence $1, 6, 35, 210, 1302$, . . .

CONCLUSION

It is hereby recommended that the polarity and its idempotent, nilpotent of partial and partial one – one signed transformation semigroups can also be study.

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