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SOME RESULTS IN PRIME RINGS WITH GENERALIZED REVERSE DERIVATIONS INVOLVING JORDAN INVOLUTION

V. S. V. Krishna Murty*[1], C. Jaya Subba Reddy[2], A. Vidhyullatha[3] and D. Dheeraj[4]

[1]Research Scholar, [2]Department of Mathematics,
S. V. University, Tirupati- 517502, Andhra Pradesh, India.

[3]Department of Mathematics,
S. P. W. Degree & PG College TTD, Tirupati - 517502, Andhra Pradesh, India.

[4]Department of Electronics and Communication Engineering,
Amrita School of Engineering, Kasavanahalli, Bangalore, India.

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ABSTRACT

Let R be a 2-torsion free prime ring and F be a generalized reverse derivation associated with a reverse derivation d of R. In this paper we establish some results regarding the commutativity of the ring R with Jordan involution.

Key Words: *Reverse derivation, Generalized reverse derivation, Involution, Jordan involution, Reverse left multiplier.*

1. INTRODUCTION

Study of the structures of rings with certain types of algebraic conditions is of considerable interest among the algebraists. The concept of reverse derivation was instigated by Herstein in [7], when he was going through Jordan derivations in prime rings. Bresar and Vukman [9] have introduced the notion of a reverse derivation. Samman *et.al* [10] and Jaya Subba Reddy *et.al* [4 & 5] have proved some important results on reverse derivations in prime and semiprime rings. Later on, these studies are extended to generalized reverse derivations of prime rings. Several authors have proved some results on the commutativity of prime and semiprime rings admitting certain type of derivations with some conditions. Huang [12] proved some theorems on commutativity of prime rings with generalized reverse derivations. Further, investigation is turned on to rings with involutions and many authors have studied derivations in the context of prime and semiprime rings with involutions (viz., [1], [2], [8] and [9]). In 1991, Yood [3] introduced the concept of Jordan involution. Zemzami [11], proved some results on commutativity of prime rings with generalized derivations involving Jordan involutions. Recently, Jaya Subba Reddy [6] have proved certain commutativity theorems in rings with involutions involving generalized reverse derivations. In this paper, we study some results in prime rings with Jordan involution satisfying certain differential identities involving generalized reverse derivations.

2. PRELIMINARIES:

Throughout this paper R denote an associative ring with centre $Z(R)$. The symbol $[u,v] = uv - vu$ denote the commutator, $\forall\, u,v \in R$ and the anti-commutator is denoted by $u \circ v = uv + vu$, $\forall\, u,v \in R$. A ring R is 2-torsion free if $2u = 0$ yields $u = 0$. A ring R is said to be prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive map $d: R \longrightarrow R$ is said to be a reverse derivation if $d(uv) = d(v)u + v\,d(u)$, for all $u,v \in R$. An additive mapping $F: R \longrightarrow R$ is said to be a generalized reverse derivation if it satisfies $F(uv) = F(v)u + vd(u)$, where $d: R \longrightarrow R$ is an associated reverse derivation. An additive map $*: R \longrightarrow R$ is said to be an involution if it satisfies the conditions (i) $(uv)^* = v^*u^*$ (ii)$(u^*)^* = u$, for all $u,v \in R$. An additive map $*: R \longrightarrow R$ is said to be a Jordan involution if it satisfies the conditions (i) $(u^*)^* = u$ (ii) $(uv + vu)^* = u^*v^* + v^*u^*$, for all $u,v \in R$. Obviously every involution is a Jordan involution but the converse is not true. A ring equipped with involution is called a ring with involution or a *- ring. In a *- ring, an element 'u' is said to be Hermitian if $u^* = u$ and is said to be Skew–Hermitian if $u^* = -u$. An additive mapping $T: R \longrightarrow R$ is said to be reverse left (resp. right) multiplier if $T(uv) = T(v)u$ (resp. $T(uv) = vT(u)$) holds for all $u, v \in R$.

Corresponding Author: V. S. V. Krishna Murty*[1],
[1]Research Scholar, Department of Mathematics,
S. V. University, Tirupati- 517502, Andhra Pradesh, India.

Fact 1: Let $(R,*)$ be a ring with Jordan involution. If R is prime and $d(h) = 0$ where $h \in Z(R) \cap H(R) - \{0\}$, then $d(s) = 0$, where $s \in Z(R) \cap S(R) - \{0\}$.

Fact 2: If a prime ring R contains a nonzero commutative right ideal I then R is commutative.

Lemma 2.1: Let I be a nonzero ideal of the prime ring R. If d is a nonzero reverse derivation of R, then d is nonzero on I. (In other words if d is zero in I then d is zero in R)

Proof: Let $d(u) = 0, \ \forall\, u \in I.$ (2.1)

Replacing u by ur in (2.1) and using hypothesis, we get $d(r)u = 0, \ \forall\, u \in I, r \in R$.

Thus $d(r)\, I = 0, \ \forall\, r \in R$. Since $I \neq \{0\}$, we get $d(r) = 0, \ \forall\, r \in R$.

Lemma 2.2: Let R be a 2-torsion free prime ring and I be a nonzero ideal of R. If R admits a reverse derivation d such that $d^2(u) = 0, \ \forall\, u \in I$, then $d = 0$.

Proof: By the hypothesis $d^2(u) = 0, \ \forall\, u \in I$ (2.2)

Replacing u by vu in (2.2) and using hypothesis, we get $2d(v)d(u) = 0, \ \forall\, u, v \in I$.

Since R is 2-torsion free, we get $d(v)d(u) = 0, \ \forall\, u, v \in I$ (2.3)

Replacing u by ru in the equation (2.3), we have $d(v)ud(r) = 0, \ \forall\, u, v \in I, r \in R$.

Again replacing u by ur in the above equation, we have $d(v)urd(r) = 0, \ \forall\, u, v \in I, r \in R$ and hence $d(v)IRd(r) = \{0\}$, $\forall\, v \in I, r \in R$.

By the primeness of R, we get $d(v)I = \{0\}$ or $d(r) = 0 \ \ \forall\, v \in I, r \in R$.

Suppose $d(v)I = \{0\}, \forall\, v \in I$. Implies that $d(v)RI = \{0\}, \forall\, v \in I$.

Since $I \neq \{0\}$ and by the primeness of R, we obtain that $d = 0$.

Lemma 2.3: Let R be a 2 torsion free prime ring and I a nonzero ideal of R. If R admits a reverse derivation d such that $[u, d(u)]$ is central, for all $u \in I$, then R is commutative.

Proof: For any arbitrary $u \in I$, we have $[u, d(u)] \in Z(R)$ (2.4)

That is $[u^2, d(u^2)] \in Z(R), \forall\, u \in I$. Implies that $[u^2, d(u)u + ud(u)] \in Z(R), \forall\, u \in I$.

Above equation can be written as $[u^2, 2ud(u) - [u, d(u)]] \in Z\,(R), \forall\, u \in I$.

Implies that $2[u^2, ud(u)] \in Z(R), \forall\, u \in I. \Rightarrow 4(\,u^2[\,u, d(u)]) \in Z\,(R), \forall\, u \in I.$
$\Rightarrow 8\, u[u, d(u)]^2 = 0, \forall\, u \in I. \Rightarrow 8\,[u, d(u)]^3 = 0, \forall\, u \in I.$

We know that the centre of a prime ring contains no nonzero nilpotent elements, we have
$2[u, d(u)] = 0, \forall\, u \in I$. Since R is 2-torsion free, we get $[u, d(u)] = 0, \forall\, u \in I$.

Therefore I is commutative. By fact 2, we get R is commutative.

Lemma 2.4: Let R be a 2-torsion free prime ring and I a nonzero ideal of R. If R admits a generalized reverse derivation F associated with a reverse derivation d such that $d(u)oF(v) = 0, \ \forall\, u, v \in I$, then either $d = 0$ or R is commutative.

Proof: We have $d(u)oF(v) = 0, \ \forall\, u, v \in I$ (2.5)

Replacing v by rv in the equation (2.5), we get
$$(d(u)ov)d(r) - v[d(u), d(r)] - F(v)[d(u), r] = 0, \ \forall\, u, v \in I, r \in R.$$

Put $r = d(u)$ in the above equation, we get
$$(d(u)o\,v)d^2(u) - v[d(u), d^2(u)] = 0, \ \forall\, u, v \in I, r \in R \quad (2.6)$$

Replacing v by zv in the equation (2.6), and using (2.6), we get $[d(u), z]v\, d^2(u) = 0$ and hence $[d(u), z]IRd^2(u) = \{0\}$, $\forall\, u, z \in I$.

By the primeness of R, we get either $[d(u), z]I = \{0\}$ or $d^2(u) = 0,\ \forall\, u, z \in I$.

Let $I_1 = \{\, u \in I / d^2(u) = 0\}$ and $I_2 = \{u \in I / [d(u), z]I = \{0\}\}$. Then I_1 and I_2 are the additive subgroups of I whose union is I. But a group cannot be the union of two of its proper subgroups and hence $I = I_1$ and $I = I_2$.

If $I = I_1$, then $d^2(u) = 0,\ \forall\, u \in I$. By the lemma 2.2, we get $d = 0$.

If $I = I_2$, then $[d(u), z]I = \{0\},\ \forall\, u, z \in I$ and thus $[d(u), z]RI = \{0\},\ \forall\, u, z \in I$.

By the primeness of R and the fact that $I \neq \{0\}$, we get that $[d(u), z] = 0,\ \forall\, u, z \in I$ (2.7)

Replacing z by uz in (2.7), and using (2.7) we obtain $[d(u), u]z = 0,\ \forall\, u, z \in I$.

Hence $[d(u), u]RI = \{0\},\ \forall\, u \in I$. The fact that $I \neq \{0\}$, then $[d(u), u] = 0,\ \forall\, u \in I$.

By the application of the lemma 2.3, if $[d(u), u],\ \forall\, u \in I$ is central, then R is commutative.

Lemma 2.5: Let R be a 2- torsion free prime ring and I a non-zero ideal of R. If R admits a generalized reverse derivation F associated with a reverse derivation d such that $[d(u), F(v)] = 0,\ \forall\, u, v \in I$, then either d = 0 or R is commutative.

Proof: Assume that $[d(u), F(v)] = 0,\ \forall\, u, v \in I$ (2.8)

Replacing v by zv in the equation (2.8) and using (2.8), we get

$$F(v)[d(u), z] + [d(u), v]d(z) + v[d(u), d(z)] = 0, \forall\, u, v, z \in I. \quad (2.9)$$

Replacing z by d(u)z in (2.9) and using (2.9), we get

$$[d(u), v]zd^2(u) + v[d(u), z]d^2(u) + vz[d(u), d^2(u)] = 0, \forall\, u, v, z \in I. \quad (2.10)$$

Replacing v by rv for all $r \in R$ in (2.10) and using (2.10), we get $[d(u), r]vzd^2(u) = 0,\ \forall\, u, v, z \in I$ and thus $[d(u), r]RId^2(u) = \{0\}, \forall\, u \in I, z \in R$.

By following the similar steps we adopted in Lemma 2.4, we can conclude that d = 0 or R is commutative.

3. MAIN RESULTS

Theorem 3.1: Let (R, *) be a 2-torsion free prime ring with Jordan involution of the second kind and F a generalized reverse derivation associated with a reverse derivation d and $F(u)od(u^*) = 0, \forall\, u \in R$, then F is a reverse left multiplier.

Proof: Assume that $F(u)od(u^*) = 0, \forall\, u \in R$. (3.1)

By linearization, $F(u)od(v^*) + F(v)od(u^*) = 0,\ \forall\, u, v \in R$.

Replacing v by v* in the above relation, we get

$$F(u)od(v) + F(v^*)od(u^*) = 0,\ \forall\, u, v \in R. \quad (3.2)$$

Substituting v = hv, where $h \in Z(R) \cap H(R) - \{0\}$ in (3.2) and using (3.2), we get

$$\big(F(u)ov + v^*od(u^*)\big)d(h) = 0,\ \forall\, u, v \in R. \quad (3.3)$$

By the primeness of R, we have either $d(h) = 0$ or $F(u)ov + v^*od(u^*) = 0, \forall\, u, v \in R$.

Suppose that $F(u)ov + v^*od(u^*) = 0, \forall\, u, v \in R$. (3.4)

Putting $v = h$ in equation (3.4), we get $2\big(F(u) + d(u^*)\big)h = 0, \forall\, u \in R$.

Since R is 2-torsion free and $h \neq 0$, we have $F(u) + d(u^*) = 0,\ \forall\, u \in R$. (3.5)

Substitute $v = s$ in (3.4) where $s \in Z(R) \cap S(R) - \{0\}$, we get $2(F(u)s + d(u^*)s^*) = 0$ which implies that

$$2(F(u) - d(u^*))s = 0, \forall u \in R.$$

Since R is 2-torsion free and $s \neq 0$, we get $F(u) - d(u^*) = 0, \ \forall u \in R.$ (3.6)

Adding (3.5) and (3. 6), we have $F = d = 0$.

Now, suppose that $d(h) = 0$. By fact 1, we have $d(s) = 0$.

Replacing v by sv in (3.2), where $s \in Z(R) \cap S(R) - \{0\}$, we get

$$\big(F(u)od(v)\big)s + \big(F(v^*)od(u^*)\big)s^* = 0, \forall u, v \in R.$$

Implies that $\big(F(u)od(v) - F(v^*)od(u^*)\big)s = 0, \forall u, v \in R.$

Since R is prime and $s \neq 0$, we have $F(u)od(v) - F(v^*)od(u^*) = 0, \forall u, v \in R.$ (3.7)

Combining (3.2) and (3.7) and using the fact that R is 2-torsion free, we get

$$F(u)od(v) = 0, \forall u, v \in R.$$

Interchanging u and v in the above relation, we get $F(v)od(u) = 0$ and hence $d(u)oF(v) = 0, \forall u, v \in R$. By using lemma 2.4, we can conclude that R is commutative.

Also, $F(u)od(v) = 0$ gives $2F(u)d(v) = 0, \forall u, v \in R$. By the fact that R is 2-torsion free and $F \neq 0$, we get $d = 0$. Therefore $F(uv) = F(v)\, u, \forall u, v \in R$.

Hence F is a reverse left multiplier. Hence the theorem is proved.

Theorem 3.2: Let $(R, *)$ be a 2 torsion free prime ring with Jordan involution of the second kind. If R admits a generalized reverse derivation F associated with a nonzero reverse derivation d and $[F(u), d(u^*)] = 0, \forall u \in R$, then R is a commutative.

Proof: We assume that $[F(u), d(u^*)] = 0, \forall u \in R$ (3.8)

By Linearization, we get $[F(u), d(v^*)] + [F(v), d(u^*)] = 0, \forall u, v \in R.$

Replacing v by v* in the above relation, we get

$$[F(u), d(v)] + [F(v^*), d(u^*)] = 0, \forall u, v \in R \quad (3.9)$$

Replacing v by hv, where $h \in Z(R) \cap H(R) - \{0\}$ in (3.9) and using (3.9), we get

$$[F(u),d(v)]h + [F(u),v]d(h) + [F(v^*),d(u^*)]h^* + [v^*, d(u^*)]d(h^*) = 0, \forall u, v \in R.$$

$$([F(u),d(v)] + [F(v^*),d(u^*)])h + ([F(u),v] + [v^*, d(u^*)])d(h) = 0, \forall u, v \in R.$$

$$([F(u),v] + [v^*, d(u^*)])d(h) = 0, \forall u, v \in R \quad (3.10)$$

Using the primeness of R, we get $[F(u),v] + [v^*,d(u^*)] = 0$ or $d(h) = 0, \forall u, v \in R$.

Suppose that $[F(u),v] + [v^*,d(u^*)] = 0, \forall u, v \in R$.

Replacing v by vs in the above relation, where $s \in Z(R) \cap S(R) - \{0\}$, we get

$$([F(u),v] - [v^*, d(u^*)])s = 0, \forall u, v \in R$$

By using the primeness of R and $s \neq 0$ then we get

$$[F(u),v] - [v^*, d(u^*)] = 0, \forall u, v \in R. \quad (3.11)$$

Adding (3.10) and (3.11) and using the 2-torsion freeness of R, we get $[F(u),v] = 0, \forall u, v \in R$.

In particular $[F(u), u] = 0, \forall u \in R$. Hence, R is commutative.

Again suppose that $d(h) = 0, \ \forall h \in H(R) \cap Z(R) - \{0\}$.

Then by the fact 1, we get $d(s) = 0, \forall s \in S(R) \cap Z(R) - \{0\}$.

Replacing v by sv in (3.9) and using (3.11), we get

$$([F(u), d(v)] - [F(v^*), d(u^*)])\, s = 0, \forall\, u, v \in R.$$

Using the primeness of R and the fact that s is nonzero, we conclude that

$$[F(u), d(v)] - [F(v^*), d(u^*)] = 0, \forall\, u, v \in R. \quad (3.12)$$

Adding (3.9), (3.12) and using R is 2-torsion free, we get $[F(u), d(v)] = 0, \forall\, u, v \in R$.

By using lemma 2.5, we can conclude that R is commutative.

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