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NOTES ON STRONGLY-n-DING PROJECTIVE MODULES

SHUO LIU*, CHANGSHENG LI, ZONGCHENG LIU AND CHI YAN

Lanzhou Petrochemical Polytechnic,
Lanzhou 730070, Gansu, People's Republic of China.

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ABSTRACT

In this paper, strongly-n-Ding projective modules are introduced and investigated, and we get a lot of interesting properties.

Key Words: *Strongly-n-Ding projective modules; Ding projective dimension.*

1. INTRODUCTION

Throughout the paper, R is a commutative ring with identity element, and all R-module are unital. If M is any R-module, we use $pd_R(M)$ and $Dpd_R(M)$ to denote projective and Ding projective dimensions of M.

In [5], the author introduced strongly Gorenstein flat module and strongly Gorenstein flat dimension, which are defined as follows:

Definition 1.1: (5) *Let n be a positive integer. An R-module M is called strongly Gorenstein flat module (we called Ding projective module) if there is an exact sequence*

$$\mathrm{P}\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective right R-modules with $M = ker(P^0 \to P^1)$ *such that* $Hom(-,\text{flat})$ *leaves the sequence exact.*

Definition 1.2: (5) *For a right R-module M, let* $SGfd(M)$ *(we called* $Dpd(M)$*) denote the infimum of the set of n such that there exists an exact sequence* $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ *of right R-modules, where each* G_i *is a strongly Gorenstein flat and call* $SGfd(M)$ *the strongly Gorenstein flat dimension of M(we called Ding projective dimension).*

The main purpose of this paper is to study some properties of strongly-n-Ding projective modules and we get some interesting results.

2. STRONGLY-*n*-DING PROJECTIVE MODULE

In this section, we will study the properties of strongly-*n*-Ding projective modules.

Lemma 2.1: (11) *A R-module M is strongly Ding projective if and only if there exists an exact sequence* $0 \to M \to P \to M \to 0$, *where P is projective and* $\mathrm{Ext}^1_R(M,F) = 0$ *for any flat F.*

Definition 2.2: *A left R-module M is said to be strongly-n-Ding projective modules if there exists a short exact sequence of left R-module* $0 \to M \to P \to M \to 0$ *with* $pd_R(P) \leq n$ *and* $\mathrm{Ext}_R^{n+1}(M,F) = 0$ *for any flat module F.*

Proposition 2.3: *Let M be a strongly-n-Ding projective module and n be a integer. If* $0 \to N \to P_n \to \cdots \to P_1 \to M \to 0$ *is an exact sequence, where* $P_1,\cdots,P_n$ *are projective, then N is strongly Ding projective module and* $Dpd_R(K) \leq n$.

Proof: The case $n = 0$ is clear by Lemma 2.1.

Corresponding Author: Shuo Liu*,
Lanzhou Petrochemical Polytechnic, Lanzhou 730070, Gansu, People's Republic of China.

Since M is strongly-n-Ding projective module, there exists an exact sequence $0 \to M \to P \to M \to 0$ with $pd_R(P) \leq n$. Consider the projective resolution of M

$$0 \to N \to P_n \to \cdots \to P_1 \to M \to 0$$

Then there exists a R-module Q such that the diagram commutative

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q \arrow[r] \arrow[d] & P_n \oplus P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \oplus P_1 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 & 0 &
\end{tikzcd}$$

Because $pd_R(P) \leq n$, then Q is projective module, and $\mathrm{Ext}^1_R(N,K) = \mathrm{Ext}^{n+1}_R(M,K) = 0$ for any flat module F, therefore N is strongly Ding projective module by Lemma2.1, hence $Dpd_R(K) \leq n$.

Proposition 2.4: *If $(M_i)_{i\in I}$ is a family of strongly-n-Ding projective modules, then $\oplus_{i\in I} M_i$ is strongly-n-Ding projective module.*

Proof: Since M_i is strongly-n-Ding projective module, then for every i, there exist short exact sequence $0 \to M_i \to P_i \to M_i \to 0$, where $pd_R(P_i) \leq n$, and $\mathrm{Ext}^{n+1}_R(M_i,F) = 0$ for every flat R-module F. Consider the exact sequence

$$0 \to \oplus_{i\in I} M_i \to \oplus_{i\in I} P_i \to \oplus_{i\in I} M_i \to 0$$

That $pd_R(\oplus_{i\in I}P_i) \leq sup\{pd_R(P_i) \mid i \in I\}$ and for every flat R-module F, $\mathrm{Ext}^{n+1}_R(\oplus_{i\in I}M_i,F) = \prod \mathrm{Ext}^{n+1}(M_i,F) = 0$, then $\oplus_{i\in I} M_i$ is strongly-n-Ding projective module.

Theorem 2.5: *Let M be a module and n be a integer. If $Dpd_R(M) \leq n$, then M is a direct summand of a strongly-n-Ding projective module.*

Proof: The case $n = 0$ is clear by [11, Theorem 2.3]. Next we assume that $n \geq 1$, since $1 \leq Dpd_R(M) \leq n$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where G is Ding projective module and $pd_R(K) \leq n-1$. According to the definition of Ding projective module, we have the short exact sequence $0 \to G \to P \to G^0 \to 0$, where P is projective module and G^0 is Ding projective module. Then we obtain the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, "="] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & G \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & G^0 \arrow[r] \arrow[d, "\|"] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & G^0 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The short exact sequence $0 \to K \to P \to D \to 0$ shows that $pd_R(D) \leq pd_R(K)+1 \leq n$. From the left half of a complete projective resolution of M, we get an exact sequence

$$0 \to G_n \to P_n \to \cdots \to P_1 \to M \to 0$$

where $P_1,\cdots,P_n$ are projectives, and G_n is Ding projective. Putting the cokernel into this diagram, we obtain exact sequence

$$0 \to G_1 \to P_1 \to M \to 0$$
$$0 \to G_2 \to P_2 \to G_1 \to 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$0 \to Gn \to Pn \to Gn-1 \to 0$$

Then $Dpd_R(G_i) \leq n-1 \leq n$ for all $1 \leq i \leq n$. According to the projective resolution of G_n

$$\cdots \to Pn+2 \to Pn+1 \to Gn \to 0$$

we get short exact sequence $0 \to G_{i+1} \to P_{i+1} \to G_i \to 0$ for all $i \geq n$, then G_i is Ding projective module by [11,Theorem 1.15]. On the other hand, because G^0 is Ding projective, therefore

$$0 \to G^0 \to P^1 \to P^2 \to P^3 \to \cdots$$

Thus $G^i = Im((P^i \to P^{i+1})$ is Ding projective for all $i \geq n$. For all $i \geq 0$, we get short exact sequence $0 \to G^i \to P^{i+1} \to G^{i+1} \to 0$, hence, we have

$$\vdots \qquad \vdots \qquad \vdots$$
$$0 \to G^1 \to P^2 \to G^2 \to 0$$
$$0 \to G^0 \to P^1 \to G^1 \to 0$$
$$0 \to M \to D \to G^0 \to 0$$
$$0 \to G_1 \to P_1 \to M \to 0$$
$$0 \to G_2 \to P_2 \to G_1 \to 0$$
$$\vdots \qquad \vdots \qquad \vdots$$

From the above sequence, we get a short exact sequence $\cancel{0}$ $N \to Q \to N \to 0$, where $N = \oplus_{i\geq 1} G_i \oplus M \oplus_{i\geq 0} G^i$, $Q = \oplus_{i\geq 1} P_i \oplus D \oplus_{i\geq 1} P^i$, obviously, $pd_R(Q) = pd_R(D) \leq n$, $Dpd_R(N) = sup\{Dpd_R(G_i), Dpd_R(G^i), Dpd_R(M)\} \leq n$. Then N is a strongly-n-Ding projective module and M is a direct summand of N.

Proposition 2.6: *For any module M and integers n, the following are equivalent:*

(1) *M is strongly-n-Ding projective module.*

(2) *There exists a short exact sequence* $0 \to M \to Q \to M \to 0$, *where* $pd_R(Q) \leq n$, *and* $\mathrm{Ext}^i_R(M,F) = 0$ *for any module F with finite flat dimension and for all i > n.*

(3) *There exists a short exact sequence* $0 \to M \to Q \to M \to 0$, *where* $pd_R(Q) < \infty$, *and* $\mathrm{Ext}^i_R(M,F) = 0$ *for any module F with finite flat dimension and for all i > n.*

Proof: Using standard arguments, this follows immediately from the definition of strongly-n-Ding projective modules.

Proposition 2.7: *Let M be an strongly-n-Ding projective module. Then M admits a surjective homomorphism* $\phi : N \to M$, *where N is strongly Ding projective module, and* $K = ker\phi$ *satisfies* $pd_R(K) = Dpd_R(M) - 1 \leq n - 1$

Proof: Pick an exact sequence, $0 \to N' \to P_n \to \cdots \to P_1 \to M \to 0$, where $P_1,\ldots,P_n$ are projective modules and N' is strongly Ding projective module by proposition 2.3. By definition of strongly Ding projective module, hence there is an exact $0 \to N' \to Q \to \cdots \to Q \to N' \to 0$, where Q is projective module, and such that the functor $\mathrm{Hom}_R(-,F)$ leaves this sequence exact, wherever F is flat.

Thus there exists homomorphism $Q \to P_i$ for $i = 1, 2,\ldots,n$, and $N \to M$, such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r] \arrow[d, "\parallel"'] & Q \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N' \arrow[r] & P_n \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

This diagram gives a chain map between complex

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_n \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

which induces an isomorphism in homology. Its mapping cone

$$0 \to Q \to P_n \oplus Q \to \cdots \to P_1 \oplus N' \to M \to 0$$

is exact, and all the modules in it, exact for $P_1 \oplus N'$ (which is strongly Ding projective) are projective. Hence the kernel K of $\phi : P_1 \oplus N' \to M$ satisfies $pd_R(K) = Dpd_R(M) - 1 \leq n - 1$

Proposition 2.8: *Let* $0 \to N \to P \to N' \to 0$ *be an exact sequence, where P is projective R-module and* $Dpd_R(N') = n < \infty$. *Then*

(1) *If* N' *is strongly Ding projective module, then so is N.*

(2) *If* N' *is strongly-n-Ding projective module for any* $n \geq 1$, *then N is strongly (n-1)-Ding projective module and* $Dpd_R(N) = n - 1$.

Proof:

(1) It is clear.

(2) Since N' is strongly-n-Ding projective module, then there exists a short exact sequence $0 \to N' \to Q \to N' \to 0$, where $pd_R(Q) \leq n$. Because $Dpd_R(N') = n$, then $pd_R(Q) = n$ by proposition 2.3. On the other hand, we have a commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q' \arrow[r] \arrow[d] & P \oplus P \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Because P is projective module, we get that $pd_R(Q') = n-1$. Since $Dpd_R(N') = n$, we conclude $Dpd_R(N) = n - 1$, Hence N is strongly-$n - 1$-Ding projective module.

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