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DECOMPOSING OF THE PARAMETRIC SPACE-BASED ON STABILITY NOTIONS FOR VECTOR OPTIMIZATION PROBLEMS

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ABSTRACT

Real engineering problems are distinguished by the presence of incommensurable and clashing objectives. Naturally, these objectives involve many parameters which its values may be defined by experts. The aim of this paper is to decompose the parametric space in vector optimization problems (VOPs) by the weighted sum approach. Also, the basic notions of stability in convex programming problems with parameters in the objective functions are defined and analyzed qualitatively for (VOP). A numerical example is given to illustrate the developed method.

Keywords: *vector optimization problems; Stability; Efficient solutions.*

1. INTRODUCTION

Many real-world problems involve several objectives that are needed to be optimized simultaneously. This type of optimization is called multi-objective optimization problems (MOPs) or vector optimization problems (VOPs) [1].

VOP has become an important research area for both scientists and researchers. In the VOP, multi-objective functions need to be optimized simultaneously. In the case of multi-objectives, there may not necessarily existence a solution that is best with respect to all objectives because of confliction among objectives. Therefore, there usually exist a set of solutions for the multi-objective case which cannot simply be compared with each other. For such solutions, called no-dominated solutions or Pareto optimal solutions, no improvement is possible in any objective function without scarifying at least one of the other objective functions [2,3].

Researchers have classified the approaches for solving multi-objective optimization problems into three categories where the decision maker engages in the decision making process expressing his/her preferences, namely a priori, interactive and a posteriori or generation approaches. In a priori approaches, the decision maker presents his/her preferences before the solution process (e.g., weights to the objective functions). The disadvantage about the a priori methods is that the decision maker has difficulty beforehand to quantify (either by means of goals or weights) his/ her preferences. The posteriori methods generate the efficient solutions of the problem are generated, and then the decision maker engages to select the most preferred one.

One of most classic methods for solving vector optimization problems (VOP) is weighted sum method that solves various single-objective sub-problems [4]. These sub-problems are generated by considering the linear combination of the objectives. By these combinations of the weights the non-dominated solutions can be obtained.

On the other hand, converting the VOP into single objective problem (SOP) by employing user defined weights (parameters) may not be accurate enough and also can lead to a false solution. So the problem has to be solved again if an error is discovered or some factors are changed which affected these parameters. So in order to solve this difficulty and assist the decision maker about the accurate parameters, stability analysis is used. Stability analysis tells us what coefficients affect greatly the solution if they are changed and what coefficients have negligible effect on the solution.

This paper presents an algorithm for decomposing the parametric space in vector optimization problem (VOP) by using the weighted norm approach. Also, the basic notions of stability in convex programming problems with parameters in the objective functions are redefined and analyzed qualitatively for VOP.

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The rest of the paper is organized as follows. In Section 2 we describe some preliminaries of the VOP. In Section 3, the stability notions of first kind are showed. The numerical example is given in Section 4 to substantiate the proposed approach. Finally, the paper is concluded in Section 5.

2. PRELIMINARIES

2.1. Statement of a vector optimization problem (VOP)

Generally, the vector optimization problem (VOP) consisting of a number of objectives and several equality and inequality constraints can be formulated as follows:

Find a vector $\mathbf{x} = (x_1, x_2, ..., x_n) \in \Omega$ for

$$\begin{aligned} & Min \;\; f(\mathbf{x}) = \left(f_1(\mathbf{x}), f_2(\mathbf{x}), .\;.\; f_K(\mathbf{x}) \right)^T \\ & subject\;\; to \quad \mathbf{x} \in \Omega, \\ & \qquad \Omega = \left\{ \mathbf{x} \in \Re^n \mid \; g_p(\mathbf{x}) \geq 0, h_j(\mathbf{x}) = a_j^T \mathbf{x} - b_j = 0, \; x_i^L \leq x_i \leq x_i^U \right\} \\ & \qquad (p = 1, 2, ..., P), \; (j = 1, 2, ..., J), (J < n), \;\; (i = 1, 2, ..., n). \end{aligned} \tag{1}$$

where $f(\mathbf{x})$, $g_p(\mathbf{x})$ and $h_j(\mathbf{x})$ stand for the objective functions, inequality and equality constrain functions with the total number of K, P and J, respectively.

In most cases, the objectives are conflicting with each other. There is no ‘‘best’’ solution for which all objectives are optimal simultaneously. The increase of one objective will lead to the decrease of other objectives. Then, there should be a set of solutions, the so-called Pareto optimal set or Pareto front, in which one solution cannot be ‘‘dominated’’ by any other member of this set. The definitions of ‘‘domination’’ and Pareto-optimality are as follows [1].

Definition 1 (Dominance): For minimal problem, a solution $\mathbf{x}_1 \in \Omega$ ($\Omega \in \Re^n$ is the feasible region) is dominating a solution $\mathbf{x}_2 \in \Omega$ (briefly written as $\mathbf{x}_1 \succ \mathbf{x}_2$ for minimization) if and only if it is superior or equal in all objectives and at least superior in one objective. This can be expressed as:

$$\mathbf{x}_1 \succ \mathbf{x}_2 \quad \text{, if } \begin{cases} \forall\, i \in 1, 2, ..., K: \; f_i(\mathbf{x}_1) \leq f_i(\mathbf{x}_2), \\ \wedge \exists\, j \in 1, 2, ..., K: \; f_j(\mathbf{x}_1) < f_j(\mathbf{x}_2). \end{cases} \tag{2}$$

Definition 2 (Pareto-optimality): Let $\mathbf{x}_1 \in \Omega$ be an arbitrary decision vector.

(a) The decision vector $\mathbf{x}_1 \in \Omega$ is said to be non-dominated regarding the set $\Omega' \subseteq \Omega$ if and only if there is no vector $\mathbf{x}_2$ in Ω' which can dominate $\mathbf{x}_1$. Formally, $\nexists \mathbf{x}_2 \in \Omega'$, $\mathbf{x}_2 \succ \mathbf{x}_1$.

(b) The decision (parameter) vector $\mathbf{x}_1$ is called Pareto-optimal if and only if $\mathbf{x}_1$ is non-dominated regarding the whole parameter space Ω.

2.2. The weighted sum method

In this subsection the weighted sum method to deal with the VOP. In this study two objective functions are considered. The Pareto-optimal solutions of the VOP can be characterized in terms of the optimal solutions of the following nonnegative weighted sum problem:

$$\begin{aligned} \mathbf{P}(\lambda): \quad & \min \boldsymbol{F}(\mathbf{x}) = \sum_{\boldsymbol{k}=1}^{\boldsymbol{K}=2} \lambda_{\boldsymbol{k}} \; \boldsymbol{f}_{\boldsymbol{k}}(\mathbf{x}) = \lambda_1 \; \boldsymbol{f}_1(\mathbf{x}) + \lambda_2 \; \boldsymbol{f}_2(\mathbf{x}), \\ & \text{subject to} \quad \mathbf{x} \in \Omega \\ & \qquad \lambda \in \psi = \left\{ \lambda \in \boldsymbol{R}^{\boldsymbol{K}} \;\middle|\; \sum_{\boldsymbol{k}=1}^{\boldsymbol{K}} \lambda_{\boldsymbol{k}} = 1, \; \lambda_{\boldsymbol{k}} \geq 0 \right\} \end{aligned} \tag{4}$$

Bi-objectives optimization runs are conducted with different weighting vector (λ) in order to locate multiple points on the Pareto front. This method is the simplest the most straight forward way of obtaining multiple points on the Pareto-optimal front. In addition, this method some sufficient conditions should be satisfied:

1) The optimal solution of the weighting problem is unique.
2) All weights of the weighting problem are strictly positive [5]. However, the proposed algorithm is constructed to obtain the set of efficient solutions of VOP. A numerical example is given for the sake of illustration.

It is easy to see that the stability of the VOP implies the stability of the problem $\mathbf{P}(\lambda)$ for decomposed parametric space for all λ .

Let

$$\boldsymbol{E}(\lambda) = \left\{ \mathbf{x}^* \in \boldsymbol{R}^n \,\middle|\, \sum_{\boldsymbol{k}=1}^{\boldsymbol{K}} \lambda_{\boldsymbol{k}} \, \boldsymbol{f}_{\boldsymbol{k}}(\mathbf{x}^*) \right\} = \min{}_{\mathbf{x}\in\Omega} \boldsymbol{F}. \tag{5}$$

A point $\mathbf{x}^*$ is Pareto optimal solution of the VOP if there exists $0 \le \lambda_{\boldsymbol{k}} \le 1, \sum_{\boldsymbol{k}=1}^{\boldsymbol{K}} \lambda_{\boldsymbol{k}} = 1$ such that $\mathbf{x}^*$ is unique optimal solution of the problem $\mathbf{P}(\lambda)$, i.e., $\boldsymbol{E}(\lambda) = \left\{ \mathbf{x}^* \right\}$.

A point $\overline{\mathbf{x}}$ is a proper Pareto optimal solution of the VOP if and only if there exists $0 < \overline{\lambda}_{\boldsymbol{k}} < 1, \sum_{\boldsymbol{k}=1}^{\boldsymbol{K}} \overline{\lambda}_{\boldsymbol{k}} = 1$ such that $\overline{\mathbf{x}} \in \boldsymbol{E}(\overline{\lambda})$.

3. THE STABILITY SET OF THE FIRST KIND

Definition 3: Suppose that ($\mathbf{P}(\lambda)$) is solvable at $\lambda^* \in \psi$ with corresponding Pareto optimal solution $\mathbf{x}^*$. Then the stability set of the first kind of problem (1) corresponding to $\mathbf{x}^*$, denoted by $\mathbf{S}(\mathbf{x}^*)$, is defined by [6,7] as follows.

$$\mathbf{S}(\mathbf{x}^*) = \{\lambda \in \psi \mid \mathbf{x}^* \text{ is Pareto optimal solution of problem (1) }] \tag{6}$$

It is easy to see that:

1) $\mathbf{S}(\mathbf{x}^*)$ is a closed and convex set.
2) If $\mathbf{int}[\mathbf{S}(\overline{\mathbf{x}}) \cap \mathbf{S}(\mathbf{x}^*)] \neq \phi,$ then $\mathbf{S}(\overline{\mathbf{x}}) = \mathbf{S}(\mathbf{x}^*)$.

Determination of the stability set of the first kind

If a point $\mathbf{x}^* \in \Omega$ is a Pareto optimal solution of problem (1), then there exists $\lambda^* \in \psi$ such that $\mathbf{x}^*$ is a Pareto optimal solution of $\mathbf{P}(\lambda^*)$. Therefore from the stability of the problem ($\mathbf{P}(\lambda)$), it follows that there exists $\mu \in R^{P+J}, \mu \ge 0$ (i.e., the equality constrained can be transformed into inequality as $h_j(\mathbf{x}) - \varepsilon \le 0$) such that (see Mangasarian [8]).

$$\lambda^{\boldsymbol{T}} \frac{\partial \boldsymbol{F}}{\partial \mathbf{x}}(\mathbf{x}^*) + \mu^{\boldsymbol{T}} \frac{\partial \boldsymbol{g}}{\partial \mathbf{x}}(\mathbf{x}^*) = 0, \quad \boldsymbol{g}(\mathbf{x}^*) \le 0, \quad \mu^{\boldsymbol{T}} \boldsymbol{g}(\mathbf{x}^*) = 0 \tag{7}$$

where η^T stands for the transpose of the vector η .

Let the set of active constraints at $\mathbf{x}^*$ is denoted by $A(\mathbf{x}^*)$:

$$\boldsymbol{A}(\mathbf{x}^*) = \{ \boldsymbol{p} \mid \boldsymbol{g}_{\boldsymbol{p}}(\mathbf{x}^*) = 0 \}. \tag{8}$$

Then the linear independent system of equations is formulated as follows

$$\lambda^{\boldsymbol{T}} \frac{\partial \boldsymbol{F}(\mathbf{x}^*)}{\partial \mathbf{x}} + \sum_{\boldsymbol{p} \in \boldsymbol{A}(\mathbf{x}^*)} \mu_{\boldsymbol{p}} \frac{\partial \boldsymbol{g}_{\boldsymbol{n}}(\mathbf{x}^*)}{\partial \mathbf{x}} = 0. \tag{9}$$

can be written in the following matrix form:

$$\begin{bmatrix} \boldsymbol{C}' & \boldsymbol{D}' \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = 0 \tag{10}$$

Where $C' = [c'_{ai}]$ is an $s \times K$ matrix , $D' = [d'_{ai}]$ is an $s \times k$ matrix, $\lambda \in \boldsymbol{R^K}$, $\mu \in \boldsymbol{R^k}$, $\lambda \geq 0,\quad \lambda \neq 0$ and $\mu \geq 0$, where $s \leq \sum_{i=\mathbf{1}}^{n} k_i$ and k is the cardinal number of the set $A(\mathbf{x}^*)$.

Suppose $d'_{ai} = 0, i = 1,2,...,k$, $a \in I \subset \{1,2,...,s\}$, where the cardinal number of I is assumed to be equal to $s-l$. Then, we ignore for the moment these rows and consider the remaining system, which will have the form

$$\begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = 0 \tag{11}$$

Here C and D are matrices of order $l \times K$ and $l \times k$, respectively. Therefore, system (11) together with the condition $\sum_{\boldsymbol{i}=1}^{\boldsymbol{K}} \boldsymbol{C}'_{\boldsymbol{ai}} \lambda_{\boldsymbol{i}} = 0$, $\boldsymbol{a} \in \boldsymbol{I}$.

Further the two propositions are considered.

Proposition 1: If $k \geq l$ then

$$\mathbf{S}(\mathbf{x}^*) = \left\{ \lambda \in \boldsymbol{R^K} \,\middle|\, \left(\lambda^{\boldsymbol{T}} \boldsymbol{C}^{\boldsymbol{T}} \left(\boldsymbol{D}_1^{\boldsymbol{T}} \right)^{-1} \right)_{\boldsymbol{i}} \leq 0, \boldsymbol{i} = 1,2,...,\boldsymbol{l},\ \sum_{\boldsymbol{i}=1}^{\boldsymbol{K}} \boldsymbol{C}'_{\boldsymbol{ai}} \lambda_{\boldsymbol{i}} = 0 \ ,\ \boldsymbol{a} \in \boldsymbol{I} \right\}. \tag{12}$$

Proposition 2: If $k < l$ then

$$\mathbf{S}(\mathbf{x}^*) = \begin{Bmatrix} \lambda \in \boldsymbol{R^K} \mid \left(\lambda^{\boldsymbol{T}} \boldsymbol{C}_2^{\boldsymbol{T}} - \boldsymbol{C}_1^{\boldsymbol{T}} \left(\boldsymbol{D}_1^{\boldsymbol{T}} \right)^{-1} \boldsymbol{D}_2^{\boldsymbol{T}} \right)_{\boldsymbol{i}} = 0,\ \boldsymbol{i} = 1,2,...,\boldsymbol{k}-\boldsymbol{l}, \\ \left(\lambda^{\boldsymbol{T}} \boldsymbol{C}_1^{\boldsymbol{T}} \left(\boldsymbol{D}_1^{\boldsymbol{T}} \right)^{-1} \right) \leq 0,\ \boldsymbol{i} = 1,2,...,\boldsymbol{k},\quad \sum_{\boldsymbol{i}=1}^{\boldsymbol{K}} \boldsymbol{C}'_{\boldsymbol{ai}} \lambda_{\boldsymbol{i}} = 0 \ ,\ \boldsymbol{a} \in \boldsymbol{I} \end{Bmatrix} \tag{13}$$

Where $D = [D_\mathbf{1}\ \ D_\mathbf{2}]$, $D_\mathbf{1}$ and D_2 are $l \times l$ and $l \times k - l$ matrices, respectively and η_i is the element in the $i-th$ column of the row vector η .

If w is normalized by the condition $\sum_{\boldsymbol{i}=1}^{\boldsymbol{K}} \lambda_{\boldsymbol{i}} = 1$, then we can construct a routine denoted as Rout 1 for the determination of the set $\mathbf{S}(\mathbf{x}^*)$.

Algorithm

Step-1: Start with $\lambda_1 = \lambda^0 = 0$ and $\lambda_2 = 1 - \lambda^0$.

Step-2: Use the Lingo software to solve ($\mathbf{P}(\lambda^0)$) , we obtain a Pareto optimal solution $\mathbf{x}^*$ of the problem (1).

Step-3: Substituting in the Kuhn-Tuker condition, we obtain system (9).

Step-4: if $s = p + k - \mathbf{1}$, then $\mathbf{S}(\mathbf{x}^*) = \{ \boldsymbol{t} \lambda^0 \mid \boldsymbol{t} > 0 \}$.

Step-5: At the end of step 3, system (11) can be easily found. Determine the set I .

Step-6: Determine $\mathbf{S}(\mathbf{x}^*)$ as follows. If $k \geq l$ then $\mathbf{S}(\mathbf{x}^*)$ is given by Eq.(12) and If $k < l$ then $\mathbf{S}(\mathbf{x}^*)$ is given by Eq.(13).

The solution procedure is straightforward and illustrated via the numerical example in the following section.

4. NUMERICAL EXPERIMENTS

The following numerical example is considered to illustrate the notions of the stability set for parametric parameters of the VOP.

Example:

$\boldsymbol{VOP}$:

$$\underset{\Omega}{\boldsymbol{Min}} \begin{cases} \boldsymbol{f}_1(\mathbf{x}) = \boldsymbol{x}^2 + \boldsymbol{y}^2 \\ \boldsymbol{f}_2(\mathbf{x}) = \boldsymbol{x}^2 - \boldsymbol{y}^2 \end{cases}$$

$$\boldsymbol{subject\ \ to} \quad \mathbf{x} \in \Omega,$$

$$\Omega = \left\{ \mathbf{x} \in \Re^{\boldsymbol{n}} \mid \ \boldsymbol{x} + \boldsymbol{y} - 1 \leq 0; \boldsymbol{x}, \boldsymbol{y} \geq 0 \ \right\}$$

The transformation using the weighted sum method is defined as follows:

$\boldsymbol{P}(\lambda)$:

$$\underset{\Omega}{\boldsymbol{Min}}[\lambda_1(\boldsymbol{x}^2 + \boldsymbol{y}^2 + \lambda_2(\boldsymbol{x}^2 - \boldsymbol{y})\ ],$$

$$\boldsymbol{subject\ \ to}$$

$$\mathbf{x} \in \Omega$$

The procedures of the Algorithm are executed as follows:

Step-1: Start with $\lambda_1 = \lambda^0 = 0$ and $w_2 = 1 - w^0$.

Step-2: using the Lingo software for solving ($\mathbf{P}(\lambda)$) , we obtain the solution of $\mathbf{P}(\lambda)\ \mathbf{x}^* = (0,1)$

Step-3: Substituting in the Kuhn-Tuker condition, we obtain the system

$$\lambda_1 + \lambda_2 + \mu_1 = 0$$

$$\lambda_1 - \lambda_2 + \mu_1 = 0$$

Step-4: Then the system takes the form

$$\boldsymbol{C} = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} \qquad \boldsymbol{D} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore \boldsymbol{s} = \boldsymbol{k} = \boldsymbol{l} = 2$$

Step-5: the stability set of the first kind can be determined as follows

$$\boldsymbol{C}^{\boldsymbol{T}}\left(\boldsymbol{D}^{\boldsymbol{T}}\right)^{-1} = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

$$\lambda^{\boldsymbol{T}}\, \boldsymbol{C}^{\boldsymbol{T}}\left(\boldsymbol{D}^{\boldsymbol{T}}\right)^{-1} = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2\lambda_1 - \lambda_2 \leq 0 \\ 2\lambda_1 - \lambda_2 \leq 0 \end{bmatrix}$$

$$\mathbf{S}(0,1) = \left\{ \lambda \in \boldsymbol{R}^2 \mid 2\lambda_1 - \lambda_2 \leq 0,\ \lambda_1 + \lambda_2 = 1 \right\}.$$

By repeating this procedure many times and at very small step, we can cover a wide range of the parametric space.

5. CONCLUSION

vector optimization problems (VOPs) has received an increasing amount of attention during the past few years as a technique for problems that involve multiple noncommensurable objectives. This paper has shown that VOP can be reformulated as finding points parametrically by using the nonnegative weighted norm approach. In addition, an algorithm for determining the stability set of the first kind is presented. Also, we discuss the stability set of the first kind for finding the set of efficient solutions and decomposing the parametric space in VOP problems.

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