

FINITE GROUPS WITH A MAXIMAL NILPOTENT SUBGROUP

Runshi Zhang, Deqin Chen and Shitian Liu*

*School of Science, Sichuan University of Science & Engineering, Zigong, 643000, China**E-mails: zhangrs-75@suse.edu.cn, chendeqin@suse.edu.cn; liust@suse.edu.cn**(Received on: 22-10-11; Accepted on: 07-11-11)*

ABSTRACT

A solvability condition for finite groups with maximal nilpotent subgroups is given as follows. Let G be a finite group and M a maximal nilpotent subgroup of G . If P is a Sylow 2-subgroup of M and every subgroups of P of order 2 or 4 is pronormal in G , then G is solvable.

Keywords: maximal nilpotent subgroups, solvable groups, pronormal subgroups

Subject Classification (2010): 20D10, 20D20.

1. INTRODUCTION:

All groups considered are finite. Hall in [7] introduced the concept of pronormality. Criteria for pronormality were given and studied by some authors (see [12, 13] and [10, 17]). D'Anielio introduced (see[5]) the concept of dual pronormality, and gave the structure of finite groups such that the n -maximal subgroups are dual pronormal. D'Anielio introduced (see[5]) the concept of \mathbb{F} -dual pronormality, and gave the structure of finite groups such that the n -maximal subgroup is \mathbb{F} -dual pronormal. Bianchi etc. in [3] introduced the concept of H -subgroup. H -subgroups were studied by Asaad in [2], Csorgo and Herzog in[4], and Guo and Wei in [9].

A group is said to be solvable if its composition factors are all of prime order (see [8]). Let G be a finite group all of whose proper subgroups are nilpotent. Then G is soluble (see [16, 9.19]). Thompson in [18] proved that if G is a finite group with a maximal nilpotent subgroup of odd order, then G is soluble. Deskins in [6] proved that G has a maximal nilpotent subgroup of class ≤ 2 , then G is soluble. Asaad in [1] proved that if G is a finite group of odd order n in which every minimal subgroup is pronormal in G , then G is supersolvable. A nature question arises:

If $2 \nmid |G|$, certain subgroups are pronormal in G , what can be said about the structure of G ?

The key of this note is to prove the following:

Main Theorem: Let G be a finite group and M a maximal nilpotent subgroup of G . If P is a Sylow 2-subgroup of M and every subgroups of P of order 2 or 4 is pronormal in G , then G is solvable.

***Corresponding author: Shitian Liu*, *E-mail: liust@suse.edu.cn**

2. PRELIMINARIES:

The concept of pronormality is one of the most important of embedding properties and was first introduced by P. Hall in his lectures in Cambridge.

Definition 2.1[7, I, 6.1] Let G be a group and $U \leq G$. Then U is said to be pronormal in G (written $U \text{ pr } G$) if, for each $g \in G$, the subgroups U and U^g are conjugate in their join $\langle U, U^g \rangle$.

Remark 2.1 Normality implies pronormality. But the converse is not the case. For example, let $G = A_4$, the alternate group of degree 4. Obviously, the Sylow 2-subgroup P of G are pronormal in G (see[8,p13]), but not normal in G . If P is normal in G , then G must have element of order 6, which is impossible.

Lemma 2.1 [7, I, 6.3] Let U be a pronormal subgroup of a group G .

- (1) If $U \leq L \leq G$, then $U \text{ pr } L$;
- (2) If $U \leq K \triangleleft G$, then $G = N_G(U)K$; in other words, the Frattini argument applies to pronormal subgroups;
- (3) If $K \triangleleft G$, then $UK \text{ pr } G$; furthermore, $UK/K \text{ pr } G/K$ and $N_G(UK) = N_G(U)K$;
- (4) If U is subnormal in G , then $U \triangleleft G$;
- (5) $N_G(U)$ is both pronormal and self-normalizing in G .

Let $\text{rad}(G)$ denote the radical of a group G , which is the largest solvable normal subgroup of G .

Lemma 2.2 [15, Theorem] If a finite group G contains a maximal nilpotent subgroup M and a non-abelian minimal normal subgroup N and if P is the Sylow 2-subgroup of M , then $G = P$.

3. THE PROOF OF MAIN THEOREM AND ITS APPLICATIONS:

In this section, we will give the proof of the Main Theorem, some applications, and also any remarks.

The Proof of Main Theorem:

Proof: Assume that the result is false, and let G be a counterexample of minimal order. Then we have:

Step: 1 $O_{p'}(G) = 1$

If $O_{p'}(G) = 1$, Let Q be normal q -subgroup of $O_{p'}(G)$, where $q \neq p$, and so Q is normal in G since Q is characteristic in $O_{p'}(G)$. Then we consider the quotient group G/Q . Let P_2 be a subgroup of P of order 2 or 4 is pronormal in G . Then, by Lemma 2.1(c), every subgroup P_2Q/Q of PQ/Q of order 2 or 4 is pronormal in G/Q , where PQ/Q is a Sylow 2-subgroup of MQ/Q .

Then G/Q , PQ/Q satisfies the hypotheses of the theorem, and the minimal choice of G implies that G/Q is solvable. This means that G is solvable by [8, 4.1(ii) (iii), p23], a contradiction. Then $O_{p'}(G) = 1$.

Step: 2 For any maximal nilpotent subgroups M , $M/F(G)$ is solvable, where $F(G)$ is the Fitting subgroup of G . Furthermore, $\Phi(G) = F(G)$, where $\Phi(G)$ is the Frattini subgroup of G .

If $F(G) \leq M$, then $G = MF(G)$ and M are solvable since the minimal choice of G . Thus G is solvable by [8, 4.1(ii)(iii), p23], a contradiction. Then, by Step 1, $M/F(G)$ is solvable. On the other hand, if $\Phi(G) > F(G)$ then there exists a normal subgroup W such that $W \leq F(G)$ but W is not in $\Phi(G)$. And so we have G/W is soluble. And since $G/\Phi(G)$ is soluble, G is isomorphic to a subgroup of $G/\Phi(G) \times G/W$. Thus, by [16, 5.1.2], G is soluble, a contradiction. Then we have $F(G) = \Phi(G)$.

Step: 3 Let $R = \text{rad}(G)$, then $G = PR$, and P is a Sylow 2-subgroup of G .

By Lemma 2.2, we have $G = PR$, If P is not a Sylow 2-subgroup, then by the maximality of M , then $|G:M| = 2^\alpha$ for some integer α . Then there exists a subgroup M_0 such that $M \leq M_0 \leq G$ and $|M:M_0| = 2$. Since for every subgroup L of P of order 2 or 4 is pronormal in G , then L is pronormal in M_0 by Lemma 2.1(a), the minimal choice of G implies that M_0 is soluble. By hypotheses, M is nilpotent, and so, since $M \triangleleft M_0$, M_0 is nilpotent, which contradicts the maximality of M . Thus P is a Sylow 2-subgroup of G .

Step: 4 the final contradiction.

By Step 2, $R \geq \Phi(G)$, and, by [11, Lemma 2.6], $F(R)$ is the direct product of minimal normal subgroups of G which is contained in R . Let $R = \langle x_1, x_2, \dots, x_n \rangle$, where $x_i \in R$ is of order $p_i \neq 2$ (p_i is prime number of G) by Step 2. Then $F(R) = \langle x_1 \rangle \times \dots \times \langle x_n \rangle$, and $\langle x_i \rangle \text{ char } F(R) \text{ char } R$. Thus $\langle x_i \rangle \triangleleft G$. Then we have $G/\langle x_i \rangle$ is soluble by Step 1, and so G is soluble, a contradiction.

This completes the proof.

Remark: 3.1 The condition of the theorem ‘nilpotent’ can’t be removed. Let $G = A_5$, the alternate group of degree 5.

The subgroup of the Sylow 2-subgroup of order 4 is pronormal in G , but G is not soluble.

Remark: 3.2 The condition of the theorem “nilpotent” can’t be replaced by “soluble”. Let $G = A_5$, the alternate group of degree 5. Obviously, A_4 , the alternate group of degree 4, is the soluble maximal subgroup of G . And the subgroup of the Sylow 2-subgroup of order 4 is pronormal in G , but G is not soluble.

Corollary: 3.1 Let G be a finite group and M a maximal nilpotent subgroup of G . If P is a Sylow 2-subgroup of M and every subgroups of P of order 2 or 4 is normal in G , then G is solvable.

Proof: By Remark 2.1, normality must be pronormal. Then we use Main Theorem to Corollary 3.1, G is soluble. This completes the proof.

Corollary: 3.4 [14, Theorem] Let G be a finite group with a nilpotent maximal subgroup M . If P is a Sylow 2-subgroup of M and $G_2 \cap P$ is normal in P for any Sylow 2-subgroup G_2 of G . Then G is soluble.

ACKNOWLEDGEMENT:

The object is supported by the Scientific Research Fund of School of Science of SUSE (Grant Number: 09LXYB02) and NSF of SUSE (Grant Number: 2010XJKYL017). The author is very grateful for the helpful suggestions of the referee.

REFERENCES:

- [1] M. Asaad, On the supersolvability of finite groups, I, Acta Mathematica Academiae Scientiarum Hungaricae, 38(1-4) (1981), 57-59.
- [2] M. Asaad, Some results on p -nilpotence and supersolvability of finite groups, Communications in Algebra, 34(2006), 4217-4224.
- [3] M. Bianchi, A.G.B. Mauri, M. Herzog, and L. Verard, On finite solvable groups in which normality is a transitive relation, Journal of Group Theory, 3(2000), 147-156.
- [4] P. Csorgo, and M. Herzog, On supersolvable groups and the nilpotator, Communications in Algebra, 32(2)(2004), 609-620.
- [5] A. D'Anillo, and M.D. Perez-Ramos, On dual pronormal subgroups and Fitting classes, in "Groups St Andrews 2001 in Oxford, London Mathematical society Lecture Note Series 304", 1(2003), 92-100.
- [6] W. E. Deskins, A condition for solvability of a finite group, Illinois Journal of Mathematics, 5(1961), 306-311.
- [7] K. Doerk, and T. Hawkes, Finite soluble groups, Walter de Gruyter, New York, 1992.
- [8] D. Gorenstein, Finite groups 2nd, Chelsea Publishing Corporation, Rhode Island, 1980.
- [9] X. Guo, and X. Wei, The influence of H -subgroups on the structure of finite groups, Journal of Group Theory, 13(2) (2010), 267-276.
- [10] L.A. Kurdachenko, and I.Y. Subbotin, On transitivity of pronormality, Commentationes Mathematicae Universitatis Carolinae, 43(4) (2002), 589-594.
- [11] D. Li, and X. Guo, The influence of c -normality of subgroups on the structure of finite groups, Journal of Pure and Applied Algebra, 150(2000), 53-60.
- [12] A. Mann, A criterion for pronormality, Journal of the London Mathematical Society Ser 1, 44 (1969), 175-176.
- [13] T. A. Peng, Finite groups with pro-normal subgroups, Proceedings of the American Mathematical Society, 20(1969), 232-234.

- [14] J. Randolph, A solvability condition for finite groups with nilpotent maximal subgroups, Bulletin of the Australian Mathematical Society, 3(1970), 273-276.
- [15] J. Randolph, On a theorem of Thompson concerning a class of non-solvable groups, Mathematische Zeitschrift, 118(1970), 340-343.
- [16] D. J. S. Robinson, A course in the theory of groups 2nd, Springer-Verlag, New York, 2004.
- [17] M. I. Saluk, Finite groups with nonabnormal pairs of subgroups, Mathematical Notes, 31(1982), 179-182.
- [18] J. G. Thompson, Normal p -complements for finite groups, Mathematische Zeitschrift, 72(1960), 332-354.
- [19] G. J. Wood, On pronormal subgroups of finite soluble groups, Archiv der Mathematik, 25(1974), 578-585.
