International Research Journal of Pure Algebra-7(4), 2017, 522-530
Available online through www.rjpa.info ISSN 2248–9037

LINEAR PARAMETERIZED INVERSE SINGULAR VALUES PROBLEM OF ANTI-SYMMETRIC ORTHOGONAL ANTI-SYMMETRIC MATRICES

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(Received On: 17-03-17; Revised & Accepted On: 17-04-17)

ABSTRACT

In this paper, we consider the linear parameterized inverse singular values problem (LPISVP)of anti-symmetric orthogonal anti-symmetric matrices which is described as follows:

Problem I: *(LPISVP) Given anti-symmetric orthogonal anti-symmetric matrices* $A_0, A_1, \cdots, A_n \in ASOASR^{m\times n}$ *, and non-negative real numbers* $\sigma_1^* \geq \cdots \geq \sigma_n^*$ *, find values of* $c = (c_1, \cdots, c_n)^T \in R^n$ *, such that the linear parameterized singular values of the matrix* $A(c) = A_0 + c_1 A_1 + \cdots + c_n A_n$ *are precisely* $\sigma_1^*, \cdots, \sigma_n^*$.
We only discuss the singular values of anti-symmetric orthogonal anti-symmetric matrices are distinct and nonzero.

We first establish the property of anti-symmetric orthogonal anti-symmetric matrices. Based on the Newton-type iterative method, we then turn Problem I into the linear parameterized inverse singular values of anti-symmetric matrices with small sizes, and solve these problems.

Keywords: *LPISVP, anti-symmetric orthogonal anti-symmetric matrices, anti-symmetric matrices, Newton-type iterative method.*

1. INTRODUCTION

Denote the set of all $m\times n$ real anti-symmetric matrices, the set of all $m\times n$ real matrices, the set of all $n\times n$ or $m\times m$ real orthogonal matrices, the set of $n\times n$ real anti-symmetric orthogonal anti-symmetric matrices by $ASR^{m\times n}$, $R^{m\times n}$, $O(n)$ or $O(m)$, and $O(m)$, respectively.

For decades, many authors have been devoted to study the solvability conditions for the inverse eigenvalues problem, associated with different symmetric matrices, bisymmetric matrices, anti-symmetric matrices, centro-symmetric matrices, and so on. For instances, we refer to the references [1-4]. The linear parameterized inverse eigenvalue problem of bisymmetric matrices were surveyed by Shi-Fang Yuan, Qing-Wen Wang, Zhi-Ping Xiong[5]. Two numerical methods for solving inverse singular value problems were considered by Moody T. Chu [6]. In this paper, we consider (*LPISVP*) of anti-symmetric orthogonal anti-symmetric matrices.

Problem I: (*LPISVP*) Given anti-symmetric orthogonal anti-symmetric matrices $A_0, A_1, \cdots, A_n \in ASOASR^{n\times n}$,and non-negative real numbers $\sigma_1^* \geq \cdots \geq \sigma_n^*$, find values of $c = (c_1, \cdots, c_n)^T \in R^n$, such that the linear parameterized singular values of the matrix $A(c) = A_0 + c_1 A_1 + \cdots + c_n A_n$ are precisely $\sigma_1^*, \cdots, \sigma_n^*$.

In this paper, Newton-type iterative method is the main approach to solve Problem I. Establishing the property of $ASOASR^{n\times n}$ and turning Problem I into *LPISVP* of anti-symmetric matrices with small sizes, and solve these problems.

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We now briefly state the contents of this paper. In Section 2, we discuss the property of $ASOASR^{n\times n}$. In Section 3, we present the Newton-type iterative method to compute the solutions of Problem I, propose the iterative scheme and discuss the algorithm for Problem II, what's more, do the convergence analysis.

2. THE PROPERTY OF ANTI-SYMMETRIC ORTHOGONAL ANTI-SYMMETRIC MATRICES

To study Problem I, we first analyze the property of $ASOASR^{n\times n}$.

I. $A^T=-A$, $(PA)^T=-PA$, with P an arbitrary orthogonal matrix and the spectral decomposition is $P=U\begin{pmatrix} I_k & \\ & -I_{n-k}\end{pmatrix}U^T$.

II. $A=U\begin{pmatrix} A_1 & \\ & A_2\end{pmatrix}U^T$, with A_1 and A_2 respectively anti-symmetric matrices.

III. The singular value decomposes of the matrices A_1 and A_2 are $A_1=U_1\begin{pmatrix} \Sigma_1 & \\ & 0\end{pmatrix}V_1^T$ and $A_2=U_2\begin{pmatrix} \Sigma_2 & \\ & 0\end{pmatrix}V_2^T$, respectively, with $U_1\in O(n-k)$, $U_2\in O(k)$, and $\Sigma_1=diag(\sigma_1,\sigma_2,\cdots,\sigma_r)$, $\Sigma_2=diag(\mu_1,\mu_2,\cdots,\mu_s)$

Denote the diagonal matrix Σ:

$$\Sigma=\begin{pmatrix}\begin{pmatrix} \Sigma_1 & \\ & 0\end{pmatrix} & \\ & \begin{pmatrix} \Sigma_2 & \\ & 0\end{pmatrix}\end{pmatrix}$$

.

Thus we have the following results.

Theorem 1: Suppose $A\in ASOASR^{n\times n}$, the singular value decomposition of A can be expressed as follows: $A=P\Sigma Q^T$, where $P=U\begin{pmatrix} U_1 & \\ & U_2\end{pmatrix}$ and $Q=U\begin{pmatrix} V_1 & \\ & V_2\end{pmatrix}$.

The set $M_s(\Sigma)$ and A, defined by:

$$M_s(\Sigma)=\{P\Sigma Q^T\,|P\in O,Q\in O\},$$
$$\mathrm{A}=\{A(c):c\in R^n\}.$$

IV. The tangent vector to $M_s(\Sigma)$ at a point $X\in M_s(\Sigma)$ is: $T(X)=XK-HX$, with both K and H anti-symmetric matrices.

3. THE SOLUTION OF PROBLEM I

We assume all singular values $\sigma_1^*,\cdots,\sigma_n^*$ are positive and distinct.

$$A(c)=U\begin{pmatrix} A(c)_1 & \\ & A(c)_2\end{pmatrix}U^T.$$

The decompose of $A_1,A_2,\cdots,A_n$ are as follows:

$$A_1=U\begin{pmatrix} A_1^1 & \\ & A_1^2\end{pmatrix}U^T, A_2=U\begin{pmatrix} A_2^1 & \\ & A_2^2\end{pmatrix}U^T,\cdots,A_n=U\begin{pmatrix} A_n^1 & \\ & A_n^2\end{pmatrix}U^T.$$

Based on the property of anti-symmetric orthogonal anti-symmetric matrices in section 2, we only discuss all singular values are distinct and describe the related problem as follows:

Problem II: Given real matrices $A_0, A_1, \cdots, A_n \in ASOASR^{n\times n}$ and non-negative real numbers $\sigma_1^* \ge \sigma_2^* \ge \cdots \ge \sigma_{n-k}^*$ and $\mu_1^* \ge \mu_2^* \ge \cdots \ge \mu_k^*$, find values of $c = (c_1, c_2, \cdots, c_n)^T$, such that the non-negative singular values of the matrix $A(c)_1 = A_0^1 + c_1 A_1^1 + \cdots + c_n A_n^1$ are precisely $\sigma_1^*, \sigma_2^*, \cdots, \sigma_{n-k}^*$,and the non-negative singular values of the matrix $A(c)_2 = A_0^2 + \cdots + c_n A_n^2$ are precisely $\mu_1^*, \mu_2^*, \cdots \mu_k^*$.

We describe the process of solving Problem II as follows:

Given $A^{(v)} \in M_s(\Sigma)$, there exist $P^{(v)} \in O(m), Q^{(v)} \in O(n)$ such that:
$P^{(v)T} A^{(v)} Q^{(v)} = \Sigma$ and $A^{(v)} + A^{(v)} K^{(v)} - H^{(v)} K^{(v)}$ with any anti-symmetric matrices $K^{(v)}$ and $H^{(v)}$ represent tangent vectors to $M_s(\Sigma)$.

There exists $A\left(c^{(v+1)}\right) \in ASOASR^{n\times n}$ such that :

$$A^{(v)} + A^{(v)} K^{(v)} - H^{(v)} A^{(v)} = A\left(c^{(v+1)}\right) \tag{1}$$

or equivalently:

$$\Sigma + \Sigma \tilde{K}^{(v)} - \tilde{H}^{(v)} \Sigma = P^{(v)T} A\left(c^{(v+1)}\right) Q^{(v)} \tag{2}$$

where

$$\begin{cases} \tilde{K}^{(v)} = Q^{(v)T} K^{(v)} Q^{(v)} = \begin{pmatrix} V_1^{(v)T} & \\ & V_2^{(v)T} \end{pmatrix} U^T K U \begin{pmatrix} V_1^{(v)} & \\ & V_2^{(v)} \end{pmatrix} = \begin{pmatrix} \tilde{K}_1^{(v)} & \\ & \tilde{K}_2^{(v)} \end{pmatrix}, \\ \tilde{H}^{(v)} = P^{(v)T} K^{(v)} P^{(v)} = \begin{pmatrix} U_1^{(v)T} & \\ & U_2^{(v)T} \end{pmatrix} U^T K U \begin{pmatrix} U_1^{(v)} & \\ & U_2^{(v)} \end{pmatrix} = \begin{pmatrix} \tilde{H}_1^{(v)} & \\ & \tilde{H}_2^{(v)} \end{pmatrix}. \end{cases}$$

Denote $W^{(v)} = P^{(v)T} A\left(c^{(v+1)}\right) Q^{(v)}$, which $w_{ij}^{(v)} = \Sigma_{ij} + \Sigma_{ii} \tilde{K}_{ij}^{(v)} - \tilde{H}_{ij}^{(v)} \Sigma_{jj}$, for $1 \le i \le n, 1 \le j \le n$, both $\tilde{K}^{(v)}$ and $\tilde{H}^{(v)}$ are anti-symmetric matrices.Only consider the diagonal elements of (2), we have:

$$\left(\sigma_1^*, \sigma_{2,}^* \cdots \sigma_{n-k}^*, \mu_1^*, \cdots, \mu_k^*\right)^T = diag\left\{P^{(v)T} A\left(c^{(v+1)}\right) Q^{(v)}\right\} = a^* + J^{(v)} c^{(v+1)}$$

with

$$P^{(v)} = U \begin{pmatrix} U_1^{(v)} & \\ & U_2^{(v)} \end{pmatrix} \text{and} \begin{array}{l} U_1^{(v)} = \left(U_1^{1(v)}, U_1^{2(v)}, \cdots, U_1^{n-k(v)}\right), \\ U_2^{(v)} = \left(U_2^{1(v)}, U_2^{2(v)}, \cdots, U_2^{k(v)}\right). \end{array}$$

$$Q^{(v)} = U \begin{pmatrix} V_1^{(v)} & \\ & V_2^{(v)} \end{pmatrix} \text{and} \begin{cases} V_1^{(v)} = \left(V_1^{1(v)}, V_1^{2(v)}, \cdots, V_1^{n-k(v)}\right), \\ V_2^{(v)} = \left(V_2^{1(v)}, V_2^{2(v)}, \cdots, V_2^{k(v)}\right). \end{cases}$$

$$\left(a^*\right)^T = \left(U_1^{1(v)T} A_0^1 V_1^{1(v)}, \cdots, U_1^{n-k(v)T} A_0^1 V_1^{n-k(v)}, U_2^{1(v)T} A_0^2 V_2^{1(v)}, \cdots, U_2^{k(v)T} A_0^2 V_2^{k(v)}\right)$$

$$J^{(v)} = \begin{pmatrix} U_1^{1(v)T} A_1^1 V_1^{1(v)} & U_1^{1(v)T} A_2^1 V_1^{1(v)} & \cdots & U_1^{1(v)T} A_n^1 V_1^{1(v)} \\ \vdots & \vdots & \vdots & \vdots \\ U_1^{n-k(v)T} A_1^1 V_1^{n-k(v)} & U_1^{n-k(v)T} A_2^1 V_1^{n-k(v)} & \cdots & U_1^{n-k(v)T} A_n^1 V_1^{n-k(v)} \\ U_2^{1(v)T} A_1^2 V_2^{1(v)} & U_2^{1(v)T} A_2^2 V_2^{1(v)} & \cdots & U_2^{1(v)T} A_n^2 V_2^{1(v)} \\ \vdots & \vdots & \vdots & \vdots \\ U_2^{k(v)T} A_1^2 V_2^{k(v)} & U_2^{k(v)T} A_2^2 V_2^{k(v)} & \cdots & U_2^{k(v)T} A_n^2 V_2^{k(v)} \end{pmatrix}_{n\times n}$$

We can get $c^{(\nu+1)}$ by solving (3), if and only if the coefficient matrix $J^{(\nu)}$ is nonsingular matrix.

Considering the Sylvester matrix equation(2) for unknown matrices $K^{(\nu)}$ and $H^{(\nu)}$, we get:

$$\begin{cases} \Sigma_1+\Sigma_1 V_1^{(\nu)T}K_1^{(\nu)}V_1^{(\nu)T}-U_1^{(\nu)T}H_1^{(\nu)}U_1^{(\nu)}\Sigma_1=U_1^{(\nu)T}A\left(c^{(\nu+1)}\right)_1 V_1^{(\nu)}, \\ \Sigma_2+\Sigma_2 V_2^{(\nu)T}K_2^{(\nu)}V_2^{(\nu)T}-U_2^{(\nu)T}H_2^{(\nu)}U_2^{(\nu)}\Sigma_2=U_2^{(\nu)T}A\left(c^{(\nu+1)}\right)_2 V_2^{(\nu)}. \end{cases} \tag{4}$$

With $U^TK^{(\nu)}U=\begin{pmatrix} K_1^{(\nu)} & \\ & K_2^{(\nu)} \end{pmatrix}$ and $U^TH^{(\nu)}U=\begin{pmatrix} H_1^{(\nu)} & \\ & H_2^{(\nu)} \end{pmatrix}$.

or equivalently:

$$\Sigma_1+\Sigma_1\tilde{K}_1^{(\nu)}-\tilde{H}_1^{(\nu)}\Sigma_1=U_1^{(\nu)T}A\left(c^{(\nu+1)}\right)_1 V_1^{(\nu)}, \tag{5}$$

$$\Sigma_2+\Sigma_2\tilde{K}_2^{(\nu)}-\tilde{H}_2^{(\nu)}\Sigma_2=U_2^{(\nu)T}A\left(c^{(\nu+1)}\right)_2 V_2^{(\nu)}. \tag{6}$$

For convenience, we denote

$$\left(w_{ij}^{(\nu)}\right)=U_1^{(\nu)T}A\left(c^{(\nu+1)}\right)_1 V_1^{(\nu)}=U_1^{(\nu)T}\left(A_0^1+c_1^{(\nu+1)}A_1^1+\cdots+c_n^{(\nu+1)}A_n^1\right)V_1^{(\nu)}$$

From (5), because $\tilde{K}_{1ii}^{(\nu)}=\tilde{H}_{1ii}^{(\nu)}=0$, we have:

$$\begin{cases} \tilde{K}_{1(ij)}^{(\nu)}=-\tilde{K}_{1(ji)}^{(\nu)}=\dfrac{\sigma_i^* w_{ij}^{(\nu)}+\sigma_j^* w_{ji}^{(\nu)}}{\left(\sigma_i^*\right)^2-\left(\sigma_j^*\right)^2}, \\ \tilde{H}_{1(ij)}^{(\nu)}=-\tilde{H}_{1(ji)}^{(\nu)}=\dfrac{\sigma_i^* w_{ji}^{(\nu)}+\sigma_j^* w_{ij}^{(\nu)}}{\left(\sigma_i^*\right)^2-\left(\sigma_j^*\right)^2}. \end{cases}$$

Thus ,we get $\tilde{K}_1^{(\nu)}$ and $\tilde{H}_1^{(\nu)}$.

The same as to (5), we denote

$$z_{ij}^{(\nu)}=U_2^{(\nu)T}A\left(c^{(\nu+1)}\right)_2 V_2^{(\nu)}=U_2^{(\nu)T}\left(A_0^2+c_1^{(\nu+1)}A_1^2+\cdots+c_n^{(\nu+1)}A_n^2\right)V_2^{(\nu)}.$$

From (5), because $\tilde{K}_{2ii}^{(\nu)}=\tilde{H}_{2ii}^{(\nu)}=0$, we have:

$$\begin{cases} \tilde{K}_{2(ij)}^{(\nu)}=-\tilde{K}_{2(ji)}^{(\nu)}=\dfrac{\mu_i^* z_{ij}^{(\nu)}+\mu_j^* z_{ji}^{(\nu)}}{\left(\mu_i^*\right)^2-\left(\mu_j^*\right)^2}, \\ \tilde{H}_{2(ij)}^{(\nu)}=-\tilde{H}_{2(ji)}^{(\nu)}=\dfrac{\mu_i^* z_{ji}^{(\nu)}+\mu_j^* z_{ij}^{(\nu)}}{\left(\mu_i^*\right)^2-\left(\mu_j^*\right)^2}. \end{cases}$$

By now,we get $\tilde{K}_2^{(\nu)}$ and $\tilde{H}_2^{(\nu)}$.

Based on our discussion mentioned above, $\tilde{K}^{(\nu)}$ and $\tilde{H}^{(\nu)}$ have been solved, and the equations in (2) are completely solved.

We now discuss the iterative scheme.

Suppose $A^{(\nu+1)}\in M_s(\Sigma)$, $A\left(c^{(\nu+1)}\right)\in \mathrm{A}$ and the equation (1) with

$$A^{(\nu)}=U\begin{pmatrix} A_1^{(\nu)} & \\ & A_2^{(\nu)} \end{pmatrix}U^T \text{ and } A\left(c^{(\nu+1)}\right)=U\begin{pmatrix} A\left(c^{(\nu+1)}\right)_1 & \\ & A\left(c^{(\nu+1)}\right)_2 \end{pmatrix}U^T;$$

then, (1) is equivalent to

$$A_1^{(v)} + A_1^{(v)} K_1^{(v)} - H_1^{(v)} A_1^{(v)} = A\left(c^{(v+1)}\right)_1, \tag{7}$$

$$A_2^{(v)} + A_2^{(v)} K_2^{(v)} - H_2^{(v)} A_2^{(v)} = A\left(c^{(v+1)}\right)_2. \tag{8}$$

First, we define four Cayley transformations:

$$R_1 = \left(1 + \frac{H_1^{(v)}}{2}\right)\left(1 - \frac{H_1^{(v)}}{2}\right)^{-1} \text{ and } R_2 = \left(1 + \frac{H_2^{(v)}}{2}\right)\left(1 - \frac{H_2^{(v)}}{2}\right)^{-1},$$

$$S_1 = \left(1 + \frac{K_1^{(v)}}{2}\right)\left(1 - \frac{K_1^{(v)}}{2}\right)^{-1} \text{ and } S_2 = \left(1 + \frac{H_2^{(v)}}{2}\right)\left(1 - \frac{H_2^{(v)}}{2}\right)^{-1}.$$

Then, the lifted matrix on $M_s(\Sigma)$ is defined to be

$$\begin{cases} A_1^{(v+1)} \approx R_1^T A_1^{(v)} S_1, \\ A_2^{(v+1)} \approx R_2^T A_2^{(v)} S_2. \end{cases}$$

From (7) and (8), we have

$$\begin{cases} A_1^{(v+1)} \approx R_1^T \left(e^{H_1^{(v)}} A\left(c^{(v+1)}\right)_1 e^{-K_1^{(v)}}\right) S_1, \\ A_2^{(v+1)} \approx R_2^T \left(e^{H_2^{(v)}} A\left(c^{(v+1)}\right)_2 e^{-K_2^{(v)}}\right) S_2. \end{cases}$$

If $\left\|H_1^{(v)}\right\|$, $\left\|H_2^{(v)}\right\|$, $\left\|K_1^{(v)}\right\|$ and $\left\|K_2^{(v)}\right\|$ are small, then $R_1^T e^{H_1^{(v)}} \approx I_1$, $R_2^T e^{H_2^{(v)}} \approx I_2$, $S_1^T e^{-K_1^{(v)}} \approx I_3$ and $S_2^T e^{-K_2^{(v)}} \approx I_4$, what's more,

$$\begin{cases} U_1^{(v+1)} = R_1^T U_1^{(v)}, \\ U_2^{(v+1)} = R_2^T U_2^{(v)}, \end{cases} \text{and} \begin{cases} V_1^{(v+1)} = S_1^T V_1^{(v)}, \\ V_2^{(v+1)} = S_2^T V_2^{(v)}. \end{cases}$$

Thus, the orthogonal matrices $U^{(v+1)}$ and $V^{(v+1)}$ can be stated as follows:

$$\begin{cases} U^{(v+1)} = \begin{pmatrix} U_1^{(v+1)} & \\ & U_2^{(v+1)} \end{pmatrix} = \begin{pmatrix} R_1^T U_1^{(v)} & \\ & R_2^T U_2^{(v)} \end{pmatrix}, \\ V^{(v+1)} = \begin{pmatrix} V_1^{(v+1)} & \\ & V_2^{(v+1)} \end{pmatrix} = \begin{pmatrix} S_1^T V_1^{(v)} & \\ & S_2^T V_2^{(v)} \end{pmatrix}. \end{cases} \tag{9}$$

The iterative scheme is now completed. An algorithm for solving Problem II proceeds as follows.

Algorithmfor Problem II

1. Given some real matrices $A_0, A_1, \cdots, A_n \in ASOASR^{n\times n}$, the singular values are $\Sigma_1 = diag\left(\sigma_1^*, \sigma_2^*, \cdots, \sigma_{n-k}^*\right)$ and $\Sigma_2 = diag\left(\mu_1^*, \mu_2^*, \cdots, \mu_k^*\right)$, the error boundary ε, and $U^{(0)} = \begin{pmatrix} U_1^{(0)} & \\ & U_2^{(0)} \end{pmatrix}, V^{(0)} = \begin{pmatrix} V_1^{(0)} & \\ & V_2^{(0)} \end{pmatrix}$.

2. For $v = 0, 1, 2, \cdots$ until convergence, do :

a. Form the matrix $J^{(v)}$,

b. Solve $c^{(v+1)}$ according to (3) if $J^{(v)}$ is non-singular matrix.

c. Form the matrix $A\left(c^{(v+1)}\right)$ given by

$$A\left(c^{(v+1)}\right) = A_0 + c_1^{(v+1)}A_1 + \cdots + c_n^{(v+1)}A_n .$$

d.Compute the anti-symmetric matrices $K^{(v)}$ and $H^{(v)}$, according to (1).

e. Compute $U^{(v+1)}$ and $V^{(v+1)}$ by solving (9), then, we have

$$\begin{cases} \left(1+\dfrac{H_1^{(v)}}{2}\right)U_1^{(v+1)} = \left(1-\dfrac{H_1^{(v)}}{2}\right)U_1^{(v)}; \left(1+\dfrac{H_2^{(v)}}{2}\right)U_2^{(v+1)} = \left(1-\dfrac{H_2^{(v)}}{2}\right)U_2^{(v)}, \\ \left(1+\dfrac{K_1^{(v)}}{2}\right)V_1^{(v+1)} = \left(1-\dfrac{K_1^{(v)}}{2}\right)V_1^{(v)}; \left(1+\dfrac{K_2^{(v)}}{2}\right)V_2^{(v+1)} = \left(1-\dfrac{K_2^{(v)}}{2}\right)V_2^{(v)}. \end{cases}$$

f. Compute:

$\sigma^{(v+1)} = \left(\sigma_1^{(v+1)}, \sigma_2^{(v+1)}, \cdots, \sigma_{n-k}^{*}\right)$, $\mu^{(v+1)} = \left(\mu_1^{*}, \mu_2^{*}, \cdots, \mu_k^{*}\right)$ by

$$\begin{cases} \sigma_i^{(v+1)} = U_1^{(v+1)iT} A\left(c^{(v+1)}\right)_1 V_1^{(v+1)i} \quad i = 1, 2, \cdots, n-k, \\ \mu_j^{(v+1)} = U_2^{(v+1)jT} A\left(c^{(v+1)}\right)_2 V_2^{(v+1)j} \quad j = 1, 2, \cdots, k. \end{cases}$$

g.Compute $\left\|\sigma^{(v+1)} - \sigma^*\right\|^2 + \left\|\mu^{(v+1)} - \mu^*\right\|^2$, where $\begin{cases} \sigma^* = \left(\sigma_1^*, \sigma_2^*, \cdots, \sigma_{n-k}^*\right), \\ \mu^* = \left(\mu_1^*, \mu_2^*, \cdots, \mu_k^*\right). \end{cases}$

h. If $\left\|\sigma^{(v+1)} - \sigma^*\right\|^2 + \left\|\mu^{(v+1)} - \mu^*\right\|^2 < \varepsilon$, then we get $c^* = c^{(v+1)}$, stop, otherwise go to **a**.

We now discuss the convergence analysis of **Algorithm**.

Denote $A\left(c^*\right) = \hat{P}\Sigma\hat{Q}^T$, with $\hat{P} = U\begin{pmatrix} \hat{U}_1 & \\ & \hat{U}_2 \end{pmatrix}$ and $\hat{Q} = U\begin{pmatrix} \hat{V}_1 & \\ & \hat{V}_2 \end{pmatrix}$,

What's more,

$$A\left(c^*\right) = U\begin{pmatrix} A\left(c^*\right)_1 & \\ & A\left(c^*\right)_2 \end{pmatrix}U^T$$

and

$$\hat{P}\Sigma\hat{Q}^T = U\begin{pmatrix} \hat{U}_1 & \\ & \hat{U}_2 \end{pmatrix}\begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix}\begin{pmatrix} \hat{V}_1^T & \\ & \hat{V}_2^T \end{pmatrix}U^T = U\begin{pmatrix} \hat{U}_1\Sigma_1\hat{V}_1^T & \\ & \hat{U}_2\Sigma_2\hat{V}_2^T \end{pmatrix}U^T .$$

Thus, we have

$$\begin{cases} A\left(c^*\right)_1 = \hat{U}_1\Sigma_1\hat{V}_1^T, \\ A\left(c^*\right)_2 = \hat{U}_2\Sigma_2\hat{V}_2^T. \end{cases}$$

At the *v*-th stage, we define

$$\begin{cases} E_1^{(v)} = P^{(v)} - \hat{P} = U\begin{pmatrix} U_1^{(v)} - \hat{U}_1 & \\ & U_2^{(v)} - \hat{U}_2 \end{pmatrix}, \\ E_2^{(v)} = Q^{(v)} - \hat{Q} = U\begin{pmatrix} V_1^{(v)} - \hat{V}_1 & \\ & V_2^{(v)} - \hat{V}_2 \end{pmatrix}. \end{cases}$$

Lemma 2: Suppose the *LPISVP* has an exact solution at c^*, $A(c^*) = \hat{P}\Sigma\hat{Q}^T$, with $\hat{P} \in O(n)$ and $\hat{Q} \in O(n)$. For any $P \in O(n)$, $Q \in O(n)$, denote the error matrices by $E_1 = P - \hat{P}$ and $E_2 = Q - \hat{Q}$. If the matrices $P, Q \in R^{n\times n}$ satisfy $\hat{P}^T P = e^H$, $\hat{Q}^T Q = e^K$, then $\|H\| = \|E\|$, $\|K\| = \|E\|$.

Similar to the Lemma 7 in[5], we can proof the conclusions.

Proof: By

$$\begin{cases} \hat{P}^T P = \hat{P}^T\left(E + \hat{P}\right) = \hat{P}^T E + I = e^H, \\ \hat{Q}^T Q = \hat{Q}^T\left(E + \hat{Q}\right) = \hat{Q}^T E + I = e^K, \end{cases} \text{and} \begin{cases} e^H = I + H + o\left(\|H\|^2\right), \\ e^K = I + K + o\left(\|K\|^2\right). \end{cases}$$

we get: $\|H\| = \|E\|$, $\|K\| = \|E\|$.

What's more,

$$\begin{cases} E_1 = P - \hat{P} = U\begin{pmatrix} U_1 - \hat{U}_1 & \\ & U_2 - \hat{U}_2 \end{pmatrix}, \\ E_2 = Q - \hat{Q} = U\begin{pmatrix} V_1 - \hat{V}_1 & \\ & V_2 - \hat{V}_2 \end{pmatrix}. \end{cases} \tag{10}$$

from (10), we have,

$$\begin{cases} \|E_1\|^2 = \left\|U_1 - \hat{U}_1\right\|^2 + \left\|V_1 - \hat{V}_1\right\|^2 = \|E_{11}\|^2 + \|E_{12}\|^2, \\ \|E_2\|^2 = \left\|U_2 - \hat{U}_2\right\|^2 + \left\|V_2 - \hat{V}_2\right\|^2 = \|E_{21}\|^2 + \|E_{22}\|^2. \end{cases}.$$

Theorem 3: Suppose all singular values $\sigma_1^*, \sigma_2^*, \cdots, \sigma_{n-k}^*$, $\mu_1^*, \mu_2^*, \cdots, \mu_k^*$ are nonzero and distinct. Suppose also that the matrix $J^{(v)}$ is non-singular. The next iteration (9) can be defined. Furthermore, there exists $\varepsilon > 0$ such that, if $\left\|E_1^{(0)}\right\| \le \varepsilon$ and $\left\|E_2^{(0)}\right\| \le \varepsilon$, then $\begin{cases} \left\|E_1^{(v+1)}\right\| = o\left(\left\|E_1^{(v)}\right\|^2\right), \\ \left\|E_2^{(v+1)}\right\| = o\left(\left\|E_2^{(v)}\right\|^2\right). \end{cases}$.

Similar to the Theorem 8 in [5], we can proof the conclusion of Theorem 3.

Proof: The matrix $P^{(v)T}A(c^*)Q^{(v)} = \Sigma$, if the matrices $H^{(v)} = \begin{pmatrix} H_1^{(v)} & \\ & H_2^{(v)} \end{pmatrix}$ and $K^{(v)} = \begin{pmatrix} K_1^{(v)} & \\ & K_2^{(v)} \end{pmatrix}$ satisfies $\begin{cases} \hat{P}^T P^{(v)} = e^{H^{(v)}} \\ \hat{Q}^T Q^{(v)} = e^{K^{(v)}} \end{cases}$, then

$$P^{(v)T}A(c^*)Q^{(v)} = P^{(v)T}\hat{P}\Sigma\hat{Q}^T Q^{(v)} = e^{-H^{(v)}}\Sigma e^{K^{(v)}}$$

That is

$$\begin{cases} U_1^{(v)T}A(c^*)_1 V_1^{(v)} = U_1^{(v)T}\hat{U}_1\Sigma_1\hat{V}_1^T V_1^{(v)} = e^{-H_1^{(v)}}\Sigma_1 e^{K_1^{(v)}}, \\ U_2^{(v)T}A(c^*)_2 V_2^{(v)} = U_2^{(v)T}\hat{U}_2\Sigma_2\hat{V}_2^T V_2^{(v)} = e^{-H_2^{(v)}}\Sigma_2 e^{K_2^{(v)}}. \end{cases}$$

By Lemma1, we have

$$\begin{cases} \left\|H_1^{(v)}\right\| = o\left(\left\|E_{11}^{(v)}\right\|\right), \left\|H_2^{(v)}\right\| = o\left(\left\|E_{21}^{(v)}\right\|\right), \\ \left\|K_1^{(v)}\right\| = o\left(\left\|E_{12}^{(v)}\right\|\right), \left\|K_2^{(v)}\right\| = o\left(\left\|E_{22}^{(v)}\right\|\right). \end{cases}$$

We write

$$\begin{cases} H_{1*}^{(v)} = U_1^{(v)} H_1^{(v)} U_1^{(v)T}; H_{2*}^{(v)} = U_2^{(v)} H_2^{(v)} U_2^{(v)T} \\ K_{1*}^{(v)} = V_1^{(v)} K_1^{(v)} V_1^{(v)T}; K_{2*}^{(v)} = V_2^{(v)} K_2^{(v)} V_2^{(v)T} \end{cases} \tag{11}$$

then

$$\begin{cases} e^{H_{1*}^{(v)}} = U_1^{(v)} U_1^T; e^{H_{2*}^{(v)}} = U_2^{(v)} U_2^T \\ e^{K_{1*}^{(v)}} = V_1^{(v)} V_1^T; e^{K_{2*}^{(v)}} = V_2^{(v)} V_2^T \end{cases}$$

and

$$\begin{cases} H_{1*}^{(v)} = I - U_1^{(v)} \hat{U}_1^T = -E_{11}^{(v)}; K_{1*}^{(v)} = I - V_1^{(v)} \hat{V}_1^T = -E_{12}^{(v)}, \\ H_{2*}^{(v)} = I - U_2^{(v)} \hat{U}_2^T = -E_{21}^{(v)}; K_{2*}^{(v)} = I - V_2^{(v)} \hat{V}_2^T = -E_{22}^{(v)}. \end{cases}$$

Thus we have

$$\begin{cases} \left\|H_{1*}^{(v)}\right\| = o\left(\left\|E_{11}^{(v)}\right\|\right); \left\|K_{1*}^{(v)}\right\| = o\left(\left\|E_{12}^{(v)}\right\|\right), \\ \left\|H_{2*}^{(v)}\right\| = o\left(\left\|E_{21}^{(v)}\right\|\right); \left\|K_{2*}^{(v)}\right\| = o\left(\left\|E_{22}^{(v)}\right\|\right). \end{cases}$$

$$\begin{aligned} P^{(v)T} A\left(c^*\right) Q^{(v)} &= e^{-H^{(v)}} \Sigma e^{K^{(v)}} \\ &= \left(I - H^{(v)} + o\left(\left\|H^{(v)}\right\|^2\right)\right) \Sigma \left(I + K^{(v)} + o\left(\left\|K^{(v)}\right\|^2\right)\right) \\ &= \Sigma + \Sigma K^{(v)} - H^{(v)} \Sigma + o\left(\left\|E^{(v)}\right\|^2\right) \end{aligned} \tag{12}$$

The deference between (2) and (12), yields:

$$P^{(v)T} \left(A\left(c^*\right) - A\left(c^{(v+1)}\right)\right) Q^{(v)} = \Sigma \left(K^{(v)} - \tilde{K}^{(v)}\right) - \left(H^{(v)} - \tilde{H}^{(v)}\right) \Sigma + o\left(\left\|E^{(v)}\right\|^2\right) \tag{13}$$

The diagonal equations of (13) lead to the next linear system:

$$J^{(v)} \left(c^* - c^{(v+1)}\right) = o\left(\left\|E^{(v)}\right\|^2\right),$$

Thus, by order of convergence, we get

$$\left\|c^* - c^{(v+1)}\right\| = o\left(\left\|E^{(v)}\right\|^2\right).$$

From (4), we obtain

$$\begin{cases} U_1^{(v)T} \left[A\left(c^*\right)_1 - A\left(c^{(v+1)}\right)_1\right] V_1^{(v)} = \Sigma_1 \left(K_1^{(v)} - \tilde{K}_1^{(v)}\right) - \left(H_1^{(v)} - \tilde{H}_1^{(v)}\right) \Sigma_1 + o\left(\left\|E_1^{(v)}\right\|^2\right), \\ U_2^{(v)T} \left[A\left(c^*\right)_2 - A\left(c^{(v+1)}\right)_2\right] V_2^{(v)} = \Sigma_2 \left(K_2^{(v)} - \tilde{K}_2^{(v)}\right) - \left(H_2^{(v)} - \tilde{H}_2^{(v)}\right) \Sigma_2 + o\left(\left\|E_2^{(v)}\right\|^2\right). \end{cases}$$

Similarly, from the off-diagonal equations of (13), it is not difficult to see that

$$\begin{cases} \left\|K_1^{(v)} - \tilde{K}_1^{(v)}\right\| = o\left(\left\|E_1^{(v)}\right\|^2\right), \\ \left\|H_1^{(v)} - \tilde{H}_1^{(v)}\right\| = o\left(\left\|E_1^{(v)}\right\|^2\right). \end{cases}$$

Because of (11), it follows that

$$\begin{cases} \left\| K_{1*}^{(v)} - K_1^{(v)} \right\| = o\left(\left\| E_1^{(v)} \right\|^2 \right), \\ \left\| H_{1*}^{(v)} - H_1^{(v)} \right\| = o\left(\left\| E_1^{(v)} \right\|^2 \right). \end{cases}$$

Observe that:

$$\begin{cases} E_{11}^{(v+1)} = U_1^{(v+1)} - \hat{U}_1 = R_1^T U_1^{(v)} - e^{-H_{1*}^{(v)}} U_1^{(v)}, \\ E_{12}^{(v+1)} = V_1^{(v+1)} - \hat{V}_1 = S_1^T V_1^{(v)} - e^{-K_{1*}^{(v)}} V_1^{(v)}. \end{cases}$$

So, it is clear now that

$$\begin{cases} \left\| E_{11}^{(v+1)} \right\| = o\left(\left\| E_1^{(v)} \right\|^2 \right), \\ \left\| E_{12}^{(v+1)} \right\| = o\left(\left\| E_1^{(v)} \right\|^2 \right). \end{cases}$$

$$\left\| E_1^{(v+1)} \right\| = \left\| E_{11}^{(v)} \right\| + \left\| E_{12}^{(v)} \right\| = o\left(\left\| E_1^{(v)} \right\|^2 \right)$$

.

Similarly, we

$$\left\| E_2^{(v+1)} \right\| = \left\| E_{21}^{(v)} \right\| + \left\| E_{22}^{(v)} \right\| = o\left(\left\| E_2^{(v)} \right\|^2 \right).$$

Till now, we complete the process of this proof.

4. CONCLUSIONS

In this paper, we consider linear parameterized inverse singular value problem of anti-symmetric orthogonal anti-symmetric matrices. Our main work is to propose the singular value decomposition for anti-symmetric orthogonal anti-symmetric matrices, then solve the *LPISVP* of anti-symmetric orthogonal anti-symmetric matrices through Newton-type iterative method.

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Source of Support: Nil, Conflict of interest: None Declared