International Research Journal of Pure Algebra-6(9), 2016, 397-408
Available online through www.rjpa.info ISSN 2248–9037

COMMON FIXED POINTS OF GENERALIZED SYMMETRIC NONSELF CONTRACTION MAPPINGS IN METRICALLY CONVEX SPACES

P. H. KRISHNA*

Department of Mathematics,
Vignan Institute of Information and Technology, Visakhapatnam-531 173, India.

(Received On: 18-03-16; Revised & Accepted On: 20-09-16)

ABSTRACT

In this paper, we prove the existence of common fixed points for a pair of nonself- mappings in complete metrically convex metric spaces. Our results extend the results of Geeta Modi, Aravind Gupta and Varun singh [14]. Examples are provided to illustrate our results.

Keywords: *Common fixed points; metrically convex metric space.*

AMS (2010) Mathematics Subject Classification: *47H10, 54H25.*

1. INTRODUCTION AND PRELIMINARIES

The study of existence of fixed points for nonself - mappings in metrically convex spaces was initiated by Assad and Kirk [2]. Assad [1] provided sufficient conditions the existence of fixed points for nonself-mappings defined on a closed subset of complete metrically convex metric spaces satisfying Kannan-type mappings [12] which have been subsequently generalized by Khan, Pathak and Khan [13].

Definition 1.1 [2]: A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that $d(x,z) + d(z,y) = d(x,y)$.

***Lemma* 1.2 [2]:** Let K be a nonempty closed subset of a complete metrically convex metric space (X, d). If $x \in K$ and $y \notin K$ then there exists a point $z \in \partial K$ (the boundary of K) such that $d(x,z) + d(z,y) = d(x,y)$.

Definition 1.3 [9]: A pair of nonself-mapping (F,T) on a nonempty subset K of a metric space (X,d) is said to be *coincidentally commuting* if $Tx, Fx \in K$ and $Tx = Fx$ implies that $FTx = TFx$.

Definition 1.4 [5]: Let K be a nonempty subset of a metric space (X,d) and $F,T: K \to X$. The pair (F,T) is said to be *weakly commuting* if for every $x, y \in K$ with $x = Fy$ and $Ty \in K$,
$d(Tx, FTy) \leq d(Ty, Fy)$.

Notice that for $K = X$, this definition reduces to that of Sessa [15].

Definition 1.5 [6]: Let K be a nonempty subset of a metric space (X,d) and $F,T : K \to X$. The pair (F,T) is said to be *compatible* if every sequence $\{x_n\} \subseteq K$ and $\lim_{n\to\infty} d\ (Fx_n, Tx_n) = 0$ and $Tx_n \in K$ (for every $n \in N$), implied that $\lim_{n\to\infty} d\ (Ty_n, F\,Tx_n) = 0$ for every sequence $\{y_n\} \subseteq K$ such that $y_n = Fx_n, n \in N$.

In 2000, M. S. Khan, Pathak and Khan [13] proved the following existing theorem.

***Theorem* 1.6 [13]:** Let (X,d) be a complete metrically convex space and K a nonempty closed subset of X. Let $T : K \to X$ be the mapping satisfying the inequality

$d(Tx,Ty) \leq a\, max\{d(x,Tx), d(y,Ty)\} + b\,\{d(x,Ty) + d(y,Tx)\}$ for every $x, y \in K$, where a and b are non-negative reals such that
$max\left\{\frac{a+b}{1-b}, \frac{b}{1-a-b}\right\} = h > 0, \max\left\{\frac{1+a+b}{1-b}h, \frac{1+b}{1-a-b}h\right\} = h' > 0$ with $\max\{h.h'\} = h'' < 1$.

*Corresponding Author: P. H. Krishna**

Further, assume that $Tx \in K$ and $x \in \partial K$, then T has a unique fixed point in K.

In 2005, M. Imdad and Ladlay Khan [10] proved a common fixed point theorem for a pair of nonself-mappings in complete metrically convex spaces which is a special case of Theorem 3.1 of [10].

Definition 1.7 [10]: Let (X, d) be a metric space and K a nonempty subset of X. Let $F, T: K \to X$ be two nonself maps. If there exists $\phi : R^+ \to R^+$ an increasing continuous function satisfying the properties

(i) $\phi(t) = 0 \Leftrightarrow t = 0 \; and$

(ii) $\phi(2t) \leq 2\phi(t)$ and there exist $a > 0 \; and \; b \geq 0 \; with \; a + 4b < 1$

$\phi\big(d(Fx, Fy)\big) \leq a \; max \left\{\frac{1}{2} \phi\big(d(Tx, Ty)\big), \phi\big(d(Tx, Fx)\big), \; \phi\big(d(Ty, Fy)\big)\right\} + b\{\phi(d(Tx, Fy)) + \phi(d(Ty, Fx))\}$ for all distinct $x, y \in K, a, b \geq 0$ such that $a + 4b < 1$.

Then F is called a generalized T contraction mapping of K into X.

Theorem 1.8: [10] Let (X, d) be a complete metrically convex space and K a nonempty closed subset of X. If F is a generalized T contraction mapping of K into X satisfying the following:

1. $\partial K \subseteq TK, FK \cap K \subseteq TK$;
2. $Tx \in \partial K \Rightarrow Fx \in K$;
3. (F, T) is coincidentally commuting;
4. TK is closed in X.

Then F and T have a unique common fixed point.

In 2014, Geeta Modi, Aravind Gupta and Varun singh [14] proved the following theorem.

Theorem 1.9 [14]: Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $F, T : K \to X$ be two mappings satisfying the inequality

$$d(Fx, Fy) \leq (a + c)d(Fx, Tx) + b\,[max\{d(Tx, Fx), d(Tx, Ty)\} + d(Ty, Fy)] \qquad (1.9.1)$$

for all $x, y \in K$ where a, b and c are nonnegative reals such that $a + 2b + c < 1$.

Further, assume that

(i) $\partial K \subseteq TK, FK \cap K \subseteq TK$;

(ii) $Tx \in \partial K \Rightarrow Fx \in K$;

(iii) (F, T) is coincidentally commuting;

(iv) TK is closed in X.

Then F and T have a unique common fixed point.

We now introduce a more general contraction condition.

Definition 1.10: Let (X, d) be a metric space and K a nonempty closed subset of X. Let $F, T: K \to X$ be two nonself maps. F is said to be a *generalized symmetric T contraction* if there exist nonnegative real a, b, c with $a + 2b + c < 1$ satisfying the inequality

$$d(Fx, Fy) \leq \frac{a+b+c}{2}[d(Fx, Tx) + d(Fy, Ty)] + \frac{b}{2}[max\{d(Tx, Fx), d(Tx, Ty)\} + \max\{d(Ty, Fy)\ (Tx, Ty\}]] \qquad (1.1.10)$$

for all distinct $x, y \in K$.

In section 2, we prove our main results.

2. Main Results

Theorem 2.1: Let (X, d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $F, T : K \to X$ be two mappings satisfying the inequality

$$d(Fx, Fy) \leq \frac{a + b + c}{2}[d(Fx, Tx) + d(Fy, Ty)] + \frac{b}{2}\,[max\{d(Tx, Fx), d(Tx, Ty)\} + max\{d(Ty, Fy), d(Tx, Ty)]$$

for any $x, y \in K$. Where a , b and c are nonnegative reals such that $a + 2b + c < \frac{1}{2}$.

Further, assume that

(i) $\partial K \subseteq TK, FK \cap K \subseteq TK$;

(ii) $Tx \in \partial K \Rightarrow Fx \in K$;

(iii) (F, T) is coincidentally commuting;

(iv) TK is closed in X.

Then F and T have a unique common fixed point.

Proof: Let $x \in \partial K$.

Then by, (i) there exists a point $x_0 \in K$ such that $x = Tx_0$.

Since $Tx_0 = x \in \partial K$, by (ii) $Fx_0 \in K$.

Hence by (i) $Fx_0 \in FK \cap K \subseteq TK$. Then there exists $x_1 \in K$ such that $y_1 = Tx_1 = Fx_0 \in K$.

Since $y_1 = Fx_0 \in K$ then there exists a point $y_2 = Fx_1$.

If $y_2 \in K$, then $y_2 \in FK \cap K \subseteq TK$ which implies that there exists $x_2 \in K$ such that $y_2 = Tx_2$.

Otherwise if $y_2 \notin K$, then by Lemma 1.2 there exists a point $p \in \partial K$ such that

$$d(Tx_1, p) + d(p, Fx_1) = d(Tx_1, y_2). \tag{2.1.2}$$

Since $p \in \partial K$, by (i), there exists a point $x_2 \in K$ such that $p = Tx_2$ so that

$$d(Tx_1, Tx_2) + d(Tx_2, y_2) = d(Tx_1, y_2) \tag{2.1.3}$$

On continuing this process, we can find a sequence $\{x_n\}$ and $\{y_n\}$ such that

(v) $y_{n+1} = F_{xn}$

(vi) $y_n \in K \Rightarrow y_n = Tx_n$ or $y_n \notin K \Rightarrow Tx_n \in \partial K$ and

$$\mathrm{d}\big(Tx_{n-1,} Tx_n\big) + \mathrm{d}(Tx_n, y_n) = \mathrm{d}(Tx_{n-1}, y_n). \tag{2.1.4}$$

We write P = $\{Tx_i \in \{Tx_n\}/Tx_i = y_i\}$ and

$$Q = \left\{Tx_i \in \frac{\{Tx_n\}}{Tx_i} \neq y_i\right\}.$$

$If\ Q \neq \emptyset,\ let\ Tx_n \in Q \Rightarrow Tx_n \neq y_n \Rightarrow y_n \notin K$

$\Rightarrow Tx_n \in \partial K$ $(by\ (vi)) \Rightarrow Fx_n \in K$ $(by\ (ii))$ Then there exists $x_{n+1} \in K$ such that

$$Tx_{n+1} = Fx_n = y_{n+1} \Rightarrow Tx_{n+1} \in \mathrm{P}\ .$$

Thus, we have $x_n \in Q \Rightarrow Tx_{n+1} \in P$.

Therefore, any two consecutive terms of $\{Tx_n\}$ can not lie in Q.

Now, the following three cases arise for which we show that

$$d(Tx_n, Tx_{n+1}) \leq \lambda d(Tx_{n-1,}\ Tx_n) \text{ for all } n.$$

Case (I): $Tx_n, Tx_{n+1} \in P$

Case (II): $Tx_n \in P, Tx_{n+1} \in Q$ and

Case (III): $Tx_n \in Q, Tx_{n+1} \in P$ $(so\ that\ Tx_{n-1} \in P)$

Case (I): $Tx_n\ Tx_{n+1} \in P$.

By (2.1.1) we have

$$\begin{aligned}
d(Tx_n, Tx_{n+1}) &= d(Fx_{n-1}, Fx_n)\\
&\leq \left(\frac{a+b+c}{2}\right)[d(Fx_{n-1}, Tx_{n-1}) + d(Fx_{n,}, Tx_n)]\\
&+\frac{b}{2}[\max\{d(Tx_{n-1}, Fx_{n-1}), d(Tx_{n-1}, Tx_n)\} + max\{d(Tx_n, Fx_n), d(Tx_{n-1}, Tx_n)\}]\\
&= \left(\frac{a+b+c}{2}\right)[d\big(y_{n-1,}y_n\big) + d\big(y_{n+1,}y_n\big)]\\
&+\frac{b}{2}[\max\{d(y_{n-1}, y_n), d(y_{n-1}, y_n)\} + max\{d(y_n, y_{n+1}), d(y_{n-1}, y_n)\}]\\
&= \left(\frac{a+b+c}{2}\right)[d\big(y_{n-1,}y_n\big) + d\big(y_{n,} y_{n+1}\big)] + \frac{b}{2}\,\mathrm{d}(y_{n-1}, y_n)\\
&+\frac{b}{2}\max\{d(y_n, y_{n+1}), d(y_{n-1}\ y_n)\}.
\end{aligned} \tag{2.1.5}$$

Suppose that $d(y_{n-1}, y_n) < d(y_n, y_{n+1})$

Then from (2.1.5)

$$d(Tx_n, Tx_{n+1}) \leq \left(\frac{a+b+c}{2}\right)[d(y_{n-1}, y_n) + d(y_n, y_{n+1})] + \frac{b}{2}\, \mathrm{d}(y_{n-1}, y_n) + \frac{b}{2}\, \mathrm{d}(y_n, y_{n+1}),$$

which implies that

$d(Tx_n, Tx_{n+1}) \leq \left(\frac{a+2b+c}{2-(a+2b+c)}\right) d(Tx_{n-1}, Tx_n)$, a contradiction to our supposition, since $\frac{a+2b+c}{2-(a+2b+c)} < 1$.

So $\max\{d(y_n, y_{n+1}), d(y_{n-1}\ y_n)\} = d(y_{n-1}, y_n)$.

Therefore, from (2.1.5), we have

$$d(Tx_n, Tx_{n+1}) \leq \left(\frac{a+b+c}{2}\right)[d(y_{n-1}, y_n) + d(y_n, y_{n+1})] + \frac{b}{2}\, \mathrm{d}(y_{n-1}, y_n) + \frac{b}{2}\, \mathrm{d}(y_{n-1}, y_n),$$

$$= \left(\frac{a+b+c}{2} + b\right) d(Tx_{n-1}, Tx_n) + \left(\frac{a+b+c}{2}\right) d(Tx_n, Tx_{n+1}),$$

and it follows that

$d(Tx_n, Tx_{n+1}) \leq \lambda d(Tx_{n-1}, Tx_n),\ where\ \lambda \frac{a+3b+c}{2-(a+2b+c)} < 1.$

Case (II): $Tx_n \in$ P and $Tx_{n+1} \in Q$.

From 2.1.4 of (vi), we have $d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, y_{n+1}) = d(Tx_n, y_{n+1})$

So that $d(Tx_n, Tx_{n+1}) \leq d(Tx_n, y_{n+1}) = d(y_n, y_{n+1}) = d(Fx_{n-1}, Fx_n)$

$$\begin{aligned} d(y_n, y_{n+1}) = d(Fx_{n-1}, Fx_n) &\leq \left(\tfrac{a+b+c}{2}\right)[d(Fx_{n-1}, Tx_{n-1}) + d(Fx_n, Tx_n)] \\ &+ \frac{b}{2} \max\{d(Tx_{n-1}, Fx_{n-1}), d(Tx_{n-1}\ Tx_n)\} + \frac{b}{2} \max\{d(Tx_n, Fx_n), d(Tx_{n-1}\ Tx_n)\} \\ &= \left(\frac{a+b+c}{2}\right) d(y_n, Tx_{n-1}) + d(y_{n+1}, Tx_n)] + \frac{b}{2} \max\{d(Tx_{n-1}, y_{n-1}), d(Tx_{n-1}, Tx_n)\} \\ &+ \frac{b}{2} \max\{d(Tx_n, y_{n+1}), d(Tx_{n-1}, Tx_n)\} \\ &= \left(\frac{a+b+c}{2}\right) d(Tx_n, Tx_{n-1}) + d(Tx_{n+1}, Tx_n)] + \frac{b}{2} \max\{d(Tx_{n-1}, Tx_n), d(Tx_{n-1}\ Tx_n)\} \\ &+ \frac{b}{2} \max\{d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)\} \\ &= \left(\frac{a+b+c}{2}\right) d(Tx_n, Tx_{n-1}) + d(Tx_{n+1}, Tx_n)] + \frac{b}{2} \mathrm{d}(Tx_{n-1}, Tx_n) \\ &+ \frac{b}{2} \max\{d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)\} \end{aligned}$$

(2.1.6) (2.1.7)

Suppose that $d(Tx_{n-1}, Tx_n) < d(y_n, y_{n+1})$

Then, from (2.1.7) we have

$d(Fx_{n-1}, Fx_n) \leq \left(\frac{a+b+c}{2}\right)[d(Tx_{n-1}, Tx_n) + d(y_n, y_{n+1})] + \frac{b}{2}\mathrm{d}(Tx_{n-1}, Tx_n) + \frac{b}{2}\mathrm{d}(y_n, y_{n+1}), and\ hence$

$\mathrm{d}(y_n, y_{n+1}) \leq \left(\frac{a+2b+c}{2-(a+2b+c)}\right)\mathrm{d}(Tx_{n-1}, Tx_n)$

$\mathrm{d}(y_n, y_{n+1}) < \mathrm{d}(Tx_{n-1}, Tx_n)$, a contradicition.

So $max\{\mathrm{d}(y_n, y_{n+1}), d(y_n, y_{n+1})\} = \mathrm{d}(Tx_{n-1}, Tx_n)$.

Hence, from (2.1.7) we have

$$\mathrm{d}(y_n, y_{n+1}) \leq \left(\frac{a+b+c}{2}\right)[d(Tx_{n-1}, Tx_n) + d(y_n, y_{n+1})] + b\mathrm{d}(Tx_{n-1}, Tx_n).$$

Therefore $\mathrm{d}(y_n, y_{n+1}) \leq \lambda d(Tx_{n-1}, Tx_n), where\ \lambda = \frac{a+3b+c}{2-(a+b+c)} < 1$

Thus from (2.1.6), we have

$d(Tx_{n-1}, Tx_n) \leq \lambda d(Tx_{n-1}, Tx_n)$.

Case (III): $Tx_n \in Q$ *and* $Tx_{n+1} \in P$. In this case, we have
$Tx_{n-1} \in P$.
$Tx_n \neq Fx_{n-1}, Tx_{n+1} = Fx_n,$ and $Tx_{n-1} = Fx_{n-2}$
$d(Tx_n, Tx_{n+1}) \leq \max\{d(Tx_{n-1}, Tx_{n+1}), d(Fx_{n-1}, Tx_{n+1})\}$

Suppose $d(Tx_n, Tx_{n+1}) \leq d(Fx_{n-1}, Tx_{n+1})$,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq d(Fx_{n-1}, Tx_{n+1}) \\ &= d(Fx_{n-1}, Fx_n) \\ &\leq \left(\frac{a+b+c}{2}\right)[d(Fx_{n-1}, Tx_{n-1}) + d(Fx_n, Tx_n)] + \frac{b}{2}\max\{d(Tx_{n-1}, Fx_{n-1}), d(Tx_{n-1}, Tx_n)\} \\ &+ \frac{b}{2}\max\{d(Tx_n, Fx_n), d(Tx_{n-1}, Tx_n)\} \\ &= \left(\frac{a+b+c}{2}\right)[d(Fx_{n-1}, Tx_{n-1}) + d(Tx_{n+1}, Tx_n)] + \frac{b}{2}\max\{d(Tx_{n-1}, Fx_{n-1}), d(Tx_{n-1}, Tx_n)\} \\ &+ \frac{b}{2}\max\{d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)\} \end{aligned}$$

Since $d(Tx_{n-1}, Fx_{n-1}) + d(Fx_{n-1}, Tx_n) = d(Tx_{n-1}, Tx_n)$
$d(Tx_{n-1}, Fx_{n-1}) \leq d(Tx_{n-1}, Tx_n)$

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq \left(\frac{a+b+c}{2}\right)[d(Tx_{n-1}, Tx_n) + d(Tx_{n+1}, Tx_n)] + \frac{b}{2}\max\{d(Tx_{n-1}, Tx_n), d(Tx_{n-1}, Tx_n)\} \\ &+ \frac{b}{2}\max\{d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)\} \end{aligned}$$

Suppose that $\max\{d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)\} = d(Tx_n, Tx_{n+1})$

$$d(Tx_n, Tx_{n+1}) \leq \left(\frac{a+b+c}{2}\right)[d(Tx_{n-1}, Tx_n) + d(Tx_{n+1}, Tx_n)] + \frac{b}{2}d(Tx_{n-1}, Tx_n) + \frac{b}{2}d(Tx_n, Tx_{n+1})$$

$$d(Tx_n, Tx_{n+1}) \leq \left(\frac{a+2b+c}{2-(a+2b+c)}\right)d(Tx_{n-1}, Tx_n)$$

$d(Tx_n, Tx_{n+1}) < d(Tx_{n-1}, Tx_n)$, a contradiction.

So $\max\{d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)\} = d(Tx_{n-1}\ Tx_n)$

$$d(Tx_n, Tx_{n+1}) \leq \left(\frac{a+b+c}{2}\right)[d(Tx_{n-1}, Tx_n) + d(Tx_{n+1}, Tx_n)] + \frac{b}{2}d(Tx_{n-1}, Tx_n) + \frac{b}{2}d(Tx_{n-1}, Tx_n)$$

$$d(Tx_n, Tx_{n+1}) \leq \left(\frac{a+3b+c}{2-(a+2b+c)}\right)d(Tx_{n-1}, Tx_n)$$

$$d(Tx_n, Tx_{n+1}) \leq \lambda d(Tx_{n-1}, Tx_n), \textit{where } \lambda = \left(\frac{a+3b+c}{2-(a+2b+c)}\right) < 1$$

On the other hand, if

$$\begin{aligned} d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_{n+1}) \text{ then } (Tx_n, Tx_{n+1}) &\leq d(Tx_{n-1}, Tx_{n+1}) \\ &\leq d(Fx_{n-2}, Fx_n) \\ &\leq \left(\frac{a+b+c}{2}\right)[d(Fx_{n-2}, Tx_{n-2}) + d(Fx_n, Tx_n)] + \frac{b}{2}\max\{d(Fx_{n-2}, Tx_{n-2}), d(Fx_n, Tx_n)\} \\ &+ \frac{b}{2}\max\{d(Tx_n, Fx_n), d(Tx_{n-2}, Tx_n)\} \\ &= \left(\frac{a+b+c}{2}\right)[d(Tx_{n-1}, Tx_{n-2}) + d(Tx_{n+1}, Tx_n)] + \frac{b}{2}\max\{d(Tx_{n-2}, Tx_{n-1}), d(Tx_{n-2}, Tx_n)\} \\ &+ \frac{b}{2}\max\{d(Tx_n, Tx_{n+1}), d(Tx_{n-2}, Tx_n)\} \end{aligned}$$

Since $d(Tx_{n-2}, Tx_n) + d(Tx_n, Tx_{n-1}) = d(Tx_{n-2}, Tx_{n-1})$

Therefore
$d(Tx_{n-2}, Tx_n) \leq d(Tx_{n-2}, Tx_{n-1})$

$$d(Tx_{n-2}, Tx_n) \leq \left(\frac{a+b+c}{2}\right)[d(Tx_{n-2}, Tx_n) + d(Tx_{n+1}, Tx_n)] + \frac{b}{2}\max\{d(Tx_{n-2}, Tx_{n-1}), d(Tx_{n-2}, Tx_{n-1})\}$$

$$+\frac{b}{2}\max\{d(Tx_n,Tx_{n+1}), d(Tx_{n-2},Tx_{n-1})\}$$
$$=\left(\frac{a+b+c}{2}\right)[d(Tx_{n-2},Tx_n)+\mathrm{d}(Tx_{n+1},Tx_n)]+\frac{b}{2}\max\{d(Tx_{n-2},Tx_{n-1}), d(Tx_{n-2},Tx_{n-1})\}$$
$$+\frac{b}{2}\max\{d(Tx_n,Tx_{n+1}), d(Tx_{n-2},Tx_{n-1})\}$$

Suppose that $\max\{d(Tx_{n-2},Tx_{n-1}), d(Tx_n,Tx_{n+1})\} = d(Tx_n,Tx_{n+1})$

$$d(Tx_n,Tx_{n+1}) \le \left(\frac{a+b+c}{2}\right)[d(Tx_{n-2},Tx_{n-1})+\mathrm{d}(Tx_n,Tx_{n+1})]+\frac{b}{2}\mathrm{d}(Tx_{n-2},Tx_{n-1})+\frac{b}{2}d(Tx_n,Tx_{n+1})$$
$$d(Tx_n,Tx_{n+1}) \le \left(\frac{a+2b+c}{2-(a+2b+c)}\right)d(Tx_{n-2},Tx_{n-1})$$

$d(Tx_n,Tx_{n+1}) < d(Tx_{n-2},Tx_{n-1})$, a contradiction.

So $\max\{d(Tx_n,Tx_{n+1}), d(Tx_{n-2},Tx_{n-1})\} = d(Tx_{n-2},Tx_{n-1})$

$$d(Tx_n,Tx_{n+1}) \le \left(\frac{a+b+c}{2}\right)[d(Tx_{n-2},Tx_{n-1})+d(Tx_{n+1},Tx_n)+\frac{b}{2}\mathrm{d}(Tx_{n-2},Tx_{n-1})+\frac{b}{2}d(Tx_{n-2},Tx_{n-1})$$
$$d(Tx_n,Tx_{n+1}) \le \left(\frac{a+3b+c}{2-(a+2b+c)}\right)d(Tx_{n-2},Tx_{n-1})$$
$$d(Tx_n,Tx_{n+1}) \le \lambda d(Tx_{n-2},Tx_{n-1}), where\ \lambda = \left(\frac{a+3b+c}{2-(a+2b+c)}\right) < 1.$$

Thus in the all case we have

$d(Tx_n,Tx_{n+1}) \le \lambda\, max\,\{d(Tx_{n-1},Tx_n), d(Tx_{n-2},Tx_{n-1})\}$

$d(Tx_{n-1},Tx_n) \le \lambda\, max\,\{d(Tx_{n-2},Tx_{n-1}), d(Tx_{n-3},Tx_{n-2})\}$.

By induction, we get

$d(Tx_n,Tx_{n+1}) \le \lambda^n max\,\{d(Tx_0,Tx_1), d(Tx_1,Tx_2)\}$.

Now for any positive integer q, we have

$$\begin{aligned}
d(Tx_n,Tx_{n+q}) &\le d(Tx_n,Tx_{n+1})+d(Tx_{n+1},Tx_{n+2})+\cdots+d(Tx_{n+q1},Tx_{n+q})\\
&\le \lambda^n max\,\{d(Tx_0,Tx_1), d(Tx_1,Tx_2)\}+\lambda^{n+1}max\,\{d(Tx_0,Tx_1), d(Tx_1,Tx_2)\}\\
&\quad+\lambda^{n+2}max\,\{d(Tx_0,Tx_1), d(Tx_1,Tx_2)\}+\ldots+\lambda^{n+q-1}max\,\{d(Tx_0,Tx_1), d(Tx_1,Tx_2)\}\\
&= \lambda^n(1+\lambda+\lambda^2+\ldots+\lambda^{q-1}max\,\{d(Tx_0,Tx_1), d(Tx_1,Tx_2)\}\\
&\le \lambda^n(1+\lambda+\lambda^2+\ldots)max\,\{d(Tx_0,Tx_1), d(Tx_1,Tx_2)\}\\
&= \lambda^n\left(\frac{1}{1-\lambda}\right)max\,\{d(Tx_0,Tx_1), d(Tx_1,Tx_2)\}d(Tx_n,Tx_{n+q}) \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Therefore $\{Tx_n\}$ is a Cauchy sequence and hence converges to a point $z\ in\ X$.

We assume that a subsequence $\{Tx_{n(k)}\}$ of $\{Tx_n\}$contained in P and TK is closed subspace of X.

Since $\{Tx_n\}$ is a Cauchy sequence in TK, it converge to a point $z \in TK$, then there exists u such that $Tu = z$ and consequently $\{Fx_{n(k)-1}\}$ and converge to z

$$(Fx_{n(k)-1},Fu) \le \left(\frac{a+b+c}{2}\right)[d(Fx_{n(k)-1},Tx_{n(k)-1})+d(Fu,Tu)]$$
$$+\frac{b}{2}\max\,\{\mathrm{d}(Tx_{n(k)-1},Fx_{n(k)-1}), d((Tx_{n(k)-1},Tu)\}+\frac{b}{2}\max\,\{\mathrm{d}(Tu,Fu), d(Tx_{n(k)-1},Tu)\}.$$

On letting $k\to\infty, we\ get$

$$\begin{aligned}
d(z,Fu) &\le \left(\frac{a+b+c}{2}\right)[d(z,z)+\ d(Fu,Tu)]+\frac{b}{2}\max\ \{d(z,z), d(z,Tu)\}+\frac{b}{2}\max\ \{d(Tu,Fu), d(z,Tu)\}.\\
&\le \left(\frac{a+b+c}{2}\right)d(Tu,Fu)+\frac{b}{2}\mathrm{d(z,Tu)}+\frac{b}{2}\max\ \{d(Tu,Fu), d(z,Tu)\}.\\
&= \left(\frac{a+b+c}{2}\right)d(Fu,Tu)+\frac{b}{2}\mathrm{d(Tu,Fu)}\\
&= \left(\frac{a+2b+c}{2}\right)d(Fu,Tu)
\end{aligned}$$

$d(z,Fu) < d(Fu,Tu) since\ (z = Tu)$.

Which gives that $Tu = Fu$ and hence u is coincidence point of F and T.

Since the pair (F, T) is coincidentally commuting,

Therefore $z = Tu = Fu$ that implies $Fz = FTu = TFu = Tz$ and hence $Fz = Tz$.

Now, we prove that z is fixed point of F.

Consider

$$\begin{aligned} d(Fz, z) &= d(Fz, Fu) \\ &\le \left(\frac{a+b+c}{2}\right)[d(Fz, Tz) + d(Fu, Tu)] + \frac{b}{2}\max\{\mathrm{d(Tz, Fz), d\ (Tz, Tu)}\} + \frac{b}{2}\max\{\mathrm{d(Tu, Fu), d\ (Tz, Tu)}\}. \\ &= \left(\frac{a+b+c}{2}\right)[d(Fz, Tz) + d(z, z)] + \frac{b}{2}\max\{\mathrm{d(Fz, Fz), d\ (Tz, z)}\} \\ &= \frac{b}{2}\mathrm{d(Fz, z)} + \frac{b}{2}\mathrm{d(Fz, z)} \end{aligned}$$

Therefore $d\ (Fz, z) \le bd(Fz, z)$

since $b < 1$ it follows that $d(Fz, z) < d(Fz, z)$, a contradiction.

So that z is fixed point of $F\ and\ z = Fz = FTu = TFu = Tz$

Therefore z is a common fixed point of $F\ and\ T$.

Uniqueness follows from the inequality 2.1.1

Theorem 2.2: Let (X, d) be complete metrically convex metric space and K a nonempty closed subset of X. Let $F, T : K \to X$ be two mappings satisfying the inequality

$$d(Fx, Fy) \le (a + 2b + c)[d(Fy, Ty)] + 2bd(Tx.Ty) \qquad (2.2.1)$$

for all $x, y, \in K$. Where $a, b\ and\ c$ are non-negative reals such the $a + 3b + c < \frac{1}{2}$.

Further, assume that

(i) $\partial K \subseteq TK, FK \cap K \subseteq TK$;

(ii) $Tx \in \partial K \Rightarrow Fx \in K$;

(iii) (F, T) is coincidentally commuting;

(iv) TK is closed in X

Then F and T have a unique common fixed point.

Proof: As in proof of the Theorem 2.1, we can find a sequence $\{x_n\}$ satisfying

(v) $Fx_n \in K \Rightarrow \mathrm{Tx_{n+1}} = Fx_n$

(vi) $Fx_n \notin K \Rightarrow \mathrm{Tx_{n+1}} \in \partial K$ and

(vii) $d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Fx_n) = d(Tx_n. Fx_n)$ (2.2.2)

Write $P = \{Tx_i \in \{Tx_n\}/Tx_i = Fx_{i-1}\}$ and $\mathrm{Q} = \{Tx_i \in \{Tx_n\}/Tx_i \ne Fx_{i-1}\}$.

Now $Tx_n \in Q \Rightarrow Tx_n \ne Fx_{n-1} \Rightarrow Fx_{n-1} \notin K$

$\Rightarrow Tx_n \in \partial K \Rightarrow Fx_n \in K\ (by\ (ii))$

$\Rightarrow Tx_n \in P$.

Thus, we have $Tx_n \in Q \Rightarrow Tx_{n+1} \in P$.

Therefore, any two consecutive terms of $\{Tx_n\}$ can not lie in Q.

Now, we have the following three cases.

Case (I): $Tx_n, Tx_{n+1} \in P$

Case (II): $Tx \in P$, $Tx_{\mathrm{n+1}} \in Q$ and

Case (III): $Tx_n \in Q$, $Tx_{n+1} \in Q$ (so that $Tx_{n-1} \in P$

Case (I): $Tx_n, Tx_{n+1} \in P$ Then by (2.1.1)

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(Fx_{n-1}, Fx_n) \\ &\leq (a+2b+c)[d(Fx_{n-1}, Tx_{n-1}) + d(Fx_n, Tx_n)] + 2bd(Tx_{n-1}, Tx_n)\} \\ &= (a+2b+c)[d(Tx_n, Tx_{n-1}) + d(Tx_{n+1}, Tx_n)] + 2bd(Tx_{n-1}, Tx_n)\} \end{aligned}$$

$$d(Tx_n, Tx_{n+1}) \leq \left(\frac{a+4b+c}{1-(a+2b+c)}\right) d(Tx_{n-1}, Tx_n)$$

$$d(Tx_n, Tx_{n+1}) \leq \lambda_1 d(Tx_{n-1}, Tx_n), \qquad \text{where } \lambda_1 = \frac{a+4b+c}{1-(a+2b+c)} < 1$$

Case (II):
$Tx_n \in P$ *and* $Tx_{n+1} \in Q$. *Then we have* $Tx_n = Fx_{n-1}$, *and* $Tx_{n+1} \neq Fx_n$

From (vii), we have
$d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Fx_n) = d(Tx_n, Fx_n)$

So that
$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq d(Tx_n, Fx_n) \\ &= d(Fx_{n-1}, Fx_n) \qquad (2.2.3) \\ &= (a+2b+c)[d(Tx_n, Tx_{n-1}) + d(Fx_n, Fx_{n-1})] + 2bd(Tx_{n-1}, Tx_n) \\ &= (a+2b+c)[d(Tx_n, Tx_{n-1}) + d(Fx_n, Fx_{n-1})] + 2bd(Tx_{n-1}, Tx_n) \\ &= (a+4b+c)[d(Tx_n, Tx_{n-1}) + (a+2b+c)d(Fx_n, Fx_{n-1})] \end{aligned}$$

$$d(Fx_n, Fx_{n-1}) \leq \left(\frac{a+4b+c}{1-(a+2b+c)}\right) d(Tx_n, Tx_{n-1})$$

Since by (2.2.3)
$d(Tx_n, Tx_{n+1}) \leq d(Fx_{n-1}, Fx_n)$, *we have*
$d(Tx_n, Tx_{n+1}) \leq \lambda_1 d(Tx_{n-1}, Tx_n)$, where $\lambda_1 = \left(\frac{a+4b+c}{1-(a+2b+c)}\right) < 1$

***Case* (III):**
$Tx_n \in Q$ *and* $Tx_{n+1} \in P$, *then* $Tx_{n-1} \in P$ *so that we have*
$Tx_{n+1} \neq Fx_{n-1}, Tx_{n+1} = Fx_n$ *and* $Tx_{n-1} = Fx_{n-2}$
$d(Tx_n, Tx_{n+1}) \leq \max\{d(Tx_{n-1}, Tx_{n+1}), d(Fx_{n-1}, Tx_{n+1})\}$

Suppose $d(Tx_{n-1}, Tx_{n+1}) \leq d(Fx_{n-1}, Tx_{n+1})$,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq d(Fx_{n-1}, Tx_{n+1}) \\ &= d(Fx_{n-1}, Fx_n) \\ &\leq (a+2b+c)[d(Fx_{n-1}, Tx_{n-1}) + d(Fx_n, Tx_n)] + 2bd(Tx_{n-1}, Tx_n)\} \\ &\leq (a+2b+c)[d(Fx_{n-1}, Tx_{n-1}) + d(Tx_{n+1}, Tx_n)] + 2bd(Tx_{n-1}, Tx_n)\} \end{aligned}$$

Since $d(Tx_{n-1}, Fx_{n-1}) + d(Fx_{n-1}, Tx_n) = d(Tx_{n-1}, Tx_n)$

$d(Tx_{n-1}, Fx_{n-1}) \leq d(Tx_{n-1}, Tx_n)$
$d(Tx_n, Tx_{n+1}) \leq (a+2b+c)[d(Tx_{n-1}, Tx_n) + d(Tx_{n+1}, Tx_n)] + 2bd(Tx_{n-1}, Tx_n)$

$$d(Tx_n, Tx_{n+1}) \leq \left(\frac{a+4b+c}{1-(a+2b+c)}\right)[d(Tx_n, Tx_{n-1})$$

$$d(Tx_n, Tx_{n+1}) \leq \lambda_1 d(Tx_{n-1}, Tx_n), \qquad \text{where } \lambda_1 = \frac{a+4b+c}{1-(a+2b+c)} < 1$$

Now, if $d(Tx_n, Tx_{n+1}) \leq d(Tx_{n-1}, Tx_{n+1})$ then

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq d(Tx_{n-1}, Tx_{n+1}) \\ &\leq d(Fx_{n-2}, Fx_n) \\ &= (a+2b+c)[d(Fx_{n-2}, Tx_{n-2}) + d(Fx_n, Tx_n)] + 2bd(Tx_{n-2}, Tx_n) \\ &\leq (a+2b+c)[d(Tx_{n-1}, Tx_{n-2}) + d(Tx_{n+1}, Tx_n)] + 2bd(Tx_{n-2}, Tx_n)\} \end{aligned}$$

Since $d(Tx_{n-2}, Tx_n) + d(Tx_n, Tx_{n-1}) = d(Tx_{n-2}, Tx_{n-1})$.

Therefore $d(Tx_{n-2}, Tx_n) \le d(Tx_{n-2}, Tx_{n-1})$

$$d(Tx_n, Tx_{n+1}) \le (a+2b+c)[d(Tx_{n-1}, Tx_{n-2}) + d(Tx_{n+1}, Tx_n)] + 2bd(Tx_{n-2}, Tx_{n-1})$$

$$d(Tx_n, Tx_{n+1}) \le (a+4b+c)[d(Tx_{n-1}, Tx_{n-2}) + (a+2b+c)d(Tx_n, Tx_{n+1})$$

$$d(Tx_n, Tx_{n+1}) \le \left(\frac{a+4b+c}{1-(a+2b+c)}\right) d(Tx_{n-2}, Tx_{n-1})$$

$$d(Tx_n, Tx_{n+1}) \le \lambda_1 d(Tx_{n-2}, Tx_{n-1}), \qquad \text{where } \lambda_1 = \frac{a+4b+c}{1-(a+2b+c)} < 1$$

Thus in the all case we have

$$d(Tx_n, Tx_{n+1}) \le \lambda_1 \max\{d(Tx_{n-1}, Tx_n), d(Tx_{n-2}, Tx_{n-1})\}$$
$$d(Tx_{n-1}, Tx_n) \le \lambda_1 \max\{d(Tx_{n-2}, Tx_{n-1}), d(Tx_{n-3}, Tx_{n-2})\}.$$

As in the proof of theorem 2.1 we can show that $\{Tx_n\}$ is a Cauchy sequence in and hence converge to a point z in X

We assume that a subsequence $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ $contained\ in\ P\ and\ TK\ is\ a\ closed\ subset\ of\ X.$

Since$\{Tx_n\}$ $is\ a\ Cauchy\ sequence\ in\ TK, it\ converge\ to\ a\ point\ w \in TK, then\ there\ exists\ v\ such\ that\ Tv = w.$ and consequently $\{Fx_{n(k)-1}\}$ $also\ converge\ to\ w.$

$$d(Fx_{n(k)-1}, Fv) \le (a+2b+c)\left[d(Fx_{n(k)-1}, Tx_{n(k)-1}) + d(Fv, Tv)\right] + 2bd(Tx_{n(k)-1}, Tv)\}$$

On letting $k \to \infty, we\ get$

$$d(w, Fv) \le (a+2b+c)[d(w,w) + d(Fv, Tv)] + 2bd(w, Tv)\}$$
$$= (a+2b+c)d(Fv, Tv)$$

$d(w, Fv) < d(Fv, Tv) since\ (w = Tv).$

$Which\ gives\ that\ Tv = Fv\ and\ hence\ v\ is\ a\ coincidence\ point\ of\ F\ and\ T$

$Since\ the\ pair\ (F.T) is\ coincidentally\ commuting, therefore\ w = Tv = Fv\ that\ implies$

$$Fz = FTv = TFv = Tw\ and\ hence\ Fw = Tw.$$

$$Consider\ d(Fw, w) = d(Fw, Fv)$$
$$\le (a+2b+c)[d(Fw, Tw) + d(Fv, Tv)] + 2bd(Tw, Tv)$$
$$= (a+2b+c)[d(Fw, Tw) + d(w, w)] + 2bd(Tw, w)$$
$$= 2bd(Tw, w)$$

Therefore $d(Fw, w) \le 2bd(Fw, w)$

$Since\ b < 1\ it\ follows\ that\ d(Fw, w) < d(Fw, w)$, a contradiction.

$So\ that\ w\ is\ a\ fixed\ point\ of\ F\ and\ w = Fw = FTv = TFv = Tw.$

$Therefore\ w\ is\ a\ common\ fixed\ point\ of\ F\ and\ T.$

Uniqueness follows from the inequality 2.2.1

The following is an example in support of Theorem 2.2

Example 2.3: Let $X = R$ be the set of reals with the usual metric, $K = \{-3\} \cup [0.1]$

We define self maps $F\ and\ T : K \to X\ by$

$$T(x) = \begin{cases} -3x, if\ 0 \le x \le 1 \\ 1,\ if\ x = -3 \end{cases},\ F(x) \begin{cases} -\frac{2}{3}, & if\ \ 0 \le x \le 1 \\ 0, & if\ x \in \{-3,1\} \end{cases}$$

The boundary of $K\ is\ \partial K = \{-3, 0, 1\} \subseteq$ TK

$TK = [-3,0] \cup \{1\} is\ closed\ in\ R.$

$FK - \left(-\frac{1}{3}, 0\right), FK \cap K = \{0\} \subseteq TK$

$Also\ T1 = -3 \in \partial K \Rightarrow F1 = 0 \in K$
$T0 = 0 \in \partial K \Rightarrow F0 = 0 \in K$

$T(-3) = 1 \in \partial K \Rightarrow F(-3) = 0 \in K$

We now verify the inequality (2.2.1)

Case (i): $(x, y) \in [0,1)$.
$d\big(F(x), F(y)\big) = \left|\frac{x-y}{3}\right| \leq (a + 2b + c)\left[\frac{5x}{2} + \frac{5y}{2}\right] + 6b|x - y|$
$\qquad = (a + 2b + c)[d(Fx, Tx) + d(Fy, Ty)] + 2bd(Tx, Ty)$

$holds\ for\ a = \frac{1}{8}, b = \frac{1}{16}\ and\ c = \frac{1}{8}$

Case (ii): $x \in [0, 1)\ and\ y = -3$
$d\big(F(x), F(y)\big) = \frac{x}{3} \leq (a + 2b + c)\left[\frac{5x}{2} + 1\right] + 2b(1 + 3x)$
$\qquad = (a + 2b + c)[d(Fx, Tx) + d(Fy, Ty)] + 2bd(Tx, Ty)$

$holds\ for\ a = \frac{1}{8}, b = \frac{1}{16}\ and\ c = \frac{1}{8}$.

Case (iii): $x = 1\ and\ y = -3$
$d\big(F(x), F(y)\big) = 0 \leq (a + 2b + c)[1 + 1] + 2b(1 + 3x)$
$\qquad = (a + 2b + c)[d(Fx, Tx) + d(Fy, Ty)] + 2bd(Tx, Ty)$

$holds\ for\ a = \frac{1}{8}, b = \frac{1}{16}\ and\ c = \frac{1}{8}$.
which shows that the contraction condition (2.2.1) is satisfied for every distinct $x, y \in K$.

Moreover "0" is a point of coincidence as T0=F0. Also TF0=0=FT0; hence the pair (F,T) is coincidentally commuting.

Thus all the conditions of Theorem 2.2 are satisfied and "0" is the unique common fixed point of F and T.

Remark 2.4: Theorem 1.9 follows as a corollary to Theorem 2.1 replacing x with y and y with x in the inequality (1.9.1), we get the following
$d\ (Fy, Fx) \leq (a + c)d(Fy, Ty) + b[\max\{d(Ty, Fy), d(Ty, Tx)\} + d(Tx, Fx)]$ (2.4.1)

Now from inequality (1.9.1) and (2.4.1), we get the inequality (2.1.1)

In the next theorem, we replace coincidentally commuting of (F, T) and closedness of TK by weak commutativity and continuity of the map F or T respectively to prove the following.

Theorem 2.5: Let (X, d) be a complete metrically convex metric space and K a closed nonempty subset of $X.\ Let\ F, T :\ K \to X$ be two mappings satisfying the inequality
$d(Fx, Fy) \leq (a + 2b + c)d[(Fx, Tx) + d(Fy, Ty)] + 2bd(Tx, Ty)$ (2.2.1)
for all $x, y \in K.\ Where\ a, b\ and\ c\ are\ non-negative\ reals\ such\ that\ a + 3b + c < \frac{1}{2}$.
Further, assume that

(i) $\partial K \subseteq TK, FK \cap K \subseteq TK, (\partial K\ is\ the\ boundary\ of\ K)$;
(ii) $Tx \in \partial K \Rightarrow Fx \in K$;
(iii) $(F, T) is\ weakly\ commuting$;
(iv) $either\ F\ or\ T\ is\ continuous\ on\ K$.

Then F and T have a unique common fixed point.

Proof: As in proof of the Theorem 2.2, we can show that sequence $\{Tx_n\}$ is a Cauchy sequence in X, it converge to a point $z \in X$. Hence we assume that there exists a sub sequence $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ which is contained i P .

Since T is continuous, $\{TTx_{n(k)}\}$ converges to a point $T\ z$. And we have
$Fx_{n(k)-1} = Tx_{n(k)}\ and\ Tx_{n(k)-1} \in K,$

Since F and T are weakly commuting, we have
$d(TTx_{n(k)}, FTx_{n(k)-1} \leq d\big(Fx_{n(k)-1}, Tx_{n(k)-1}\big)$.

On letting $k \to \infty, we\ get$
$d(Tz, FTx_{n(k)-1}) \to 0.$
$d\big(FTx_{n(k)-1}, Fz\big) \leq (a+2b+c)[d\big(FTx_{n(k)-1}, TTx_{n(k)-1}\big) + d(Fz, Tz)] + 2bd(TTx_{n(k)-1}, Tz)\}.$

On letting $k \to \infty, we\ get$
$d(Tz, Fz) \leq (a+2b+c)[d(Tz, Tz) + d(Fz, Tz) + 2bd(Tz, Tz)\}]$
$= (a+2b+c)d(Tz, Fz)$

$d(Tz, Fz) < d(Tz, Fz)\ since\ \left(a+3b+c<\frac{1}{2}\right)$, a contradiction. Hence $Fz = Tz$.

Now we prove that $Tz = z$.

Suppose that $Tz \neq z$.
$d\big(Tx_{n(k)}, Tz\big) = d(Fx_{n(k)-1}, Fz)$
$\leq (a+2b+c)[d\big(Fx_{n(k)-1}, Tx_{n(k)-1}\big) + d(Fz, Tz)] + 2bd(TTx_{n(k)-1}, Tz)\}.$

On letting $k \to \infty, we\ get$
$d(z, Tz) \leq (a+2b+c)[d(z, z) + d(Fz, Tz)] + 2bd(z, Tz)\}]$
$= 2bd(z, Tz)$

$d(z. Tz) < d(z, Tz)$, a contradiction

Thus $z = T\ z\ = F\ z$ and hence z is a common fixed point of F and T .

Finally, we prove a theorem when weak commutativity is replaced by compatibility.

Theorem 2.6: Let (X, d) be a complete metrically convex metric space and K a closed nonempty subset of $X. Let\ F, T\colon K \to X$ be two mappings satisfying the inequality
$d(Fx, Fy) \leq\ (a+2b+c)[d(Fx, Tx) + d(Fy, Ty)] + 2bd(Tx, Ty)$ (2.2.1)
for all $x, y \in K$. Where *a, b* and *c* are non- negative reals such that $a + 3b + c < \frac{1}{2}$
Further, assume that
(i) $\partial K \subseteq TK, FK \cap K \subseteq TK, (\partial K\ is\ the\ boundary\ of\ K)$;
(ii) $Tx \in \partial K \Rightarrow Fx \in K$;
(iii) the pair $(F, T\ )$ is compatible;
(iv) either $F\ or\ T$ is continuous on K.
Then $F\ and\ T$ have a unique common fixed point.

Proof: As in proof of the Theorem 2.2, we can show that sequence $\{Tx_n\}$ is a Cauchy sequence in X, it converge to a point $z \in X$.

Hence we assume that there exists a sub sequence $\{Tx_{n(k)}\}$ of $\{Tx_n\}$

which is contained in P

And we have $Fx_{n(k)-1} = Tx_{n(k)}\ and\ Tx_{n(k)-1} \in K$

Since the pair (F, T) is compatible,

we have $\lim_{n\to\infty} d(Fx_{n(k)-1}, Tx_{n(k)-1} = 0$

that implies $\lim_{n\to\infty} d(TTx_{n(k)}, FTx_{n(k)-1} = 0.$

By continuity of T, it follows that $FTx_{n(k)-1} \to Tz\ as\ k \to \infty$.

Now as in the proof of Theorem 2.4, we can show that z is a common fixed point of $F\ and\ T$.

REFERENCES

1. N. A. Assad, On a fixed point theorem of Kannan in Banach spaces, Tamkang J. Math. 7 (1976), no.1, 9194.1
2. N. A. Assad and W. A. Kirk, Fixed point theorems for set-valued mappings of contractive type, Pacific c J. Math. 43 (1972), no. 3.
3. R. M. Bianchini, Su un problema di S. Reich riguardante la teoria dei punti ssi, Boll. Un. Mat. Ital. (4) 5 (1972), 103108
4. S. K. Chatterjea, Fixed-point theorems, C. R. Acad. Bulgare Sci. 25 (1972).
5. O. Hadzic, On coincidence points in convex metric spaces, Univ. u Novom Sadu Zb. Rad. Prirod.- Mat. Fak. Ser. Mat. 19 (1989), no. 2.
6. O. Hadzic and Lj. Gajic, Coincidence points for set-valued mappings in convex metric spaces, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser.Mat. 16 (1986), no. 1.
7. G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci. 9 (1986), no. 4.
8. M. Imdad, Remarks on fixed point theorems for nonself mappings, Aligarh Bull. Math. 22 (2003), no. 1, 1517.
9. M. Imdad and S. Kumar, Boyd and Wong type theorems for two pairs of non-self mappings, Nonlinear Anal. Forum 8 (2003), no. 1.
10. M. Imdad and Ladlay Khan, Common fixed point theorems for Nonself-mappings in metrically convex spaces via altering distances, International Journal of Mathematics and Mathematical Sciences 2005:24 (2005) 40294039 DOI: 10.1155/IJMMS.2005.4029 1, 1.7, 1.8
11. M. D. Khan and R. Bharadwaj, A fixed point theorem in metrically convex space, Indian J. Math. 43 (2001), no. 3.
12. R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968), 7176. 1
13. M. S. Khan, H. K. Pathak, and M. D. Khan, Common fixed point theorems for Nonself-mappings in metrically convex spaces, Georgian Math. J. 7 (2000), no. 3.
14. Geeta Modi, Aravind Gupta and Varun Singh, Some fixed point theorems in metrically convex spaces, International Journal of Mathematics and its Applications. 2(1), 2014, 41-48. (document), 1, 1.9
15. S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. (Beograd) (N.S.) 32(46) (1982).

Source of Support: Nil, Conflict of interest: None Declared