FUZZY VERSION OF SOFT INT G-MODULES

1G. SUBBIAH*, 2M. NAVANEETHAKRISHNAN, 3D. RADHA AND 4S. ANITHA

1*Associate Professor in Mathematics,
Sri K. G. S. Arts College, Srivaikuntam-628 619, (T.N.), India.

2Associate Professor in Mathematics, Kamaraj College,
Thoothukudi-628 003, (T.N.), India.

3Assistant Professor in Mathematics,
A. P. C. Mahalaxmi College for Women, Thoothukudi-628 002, (T.N.), India.

4Assistant Professor in Mathematics,
M. I. E. T Engineering College, Trichy-620007. (T.N.), India.

(Received On: 10-07-16; Revised & Accepted On: 28-07-16)

ABSTRACT

In this paper, we introduce fuzzy version of soft int-G-modules of a vector space with respect to soft structures, which are fuzzy soft int-G-modules (IFSG-module). These new concepts show that how a soft set effects on a G-module of a vector space in the mean of intersection, union and inclusion of sets and thus, they can be regarded as a bridge among classical sets, fuzzy soft sets and vector spaces. We then investigate their related properties with respect to soft set operations, soft image, soft pre-image, soft anti image, α-inclusion of fuzzy soft sets and linear transformations of the vector spaces. Furthermore, we give the applications of these new G-modules on vector spaces.

Index terms: Soft set, IFSG-module, fuzzy soft image, fuzzy soft anti image, trivial, whole.

1. INTRODUCTION

The concept of soft set theory is introduced by Molodtsov [1] to overcome uncertainties which cannot be dealt with by classical methods in many areas such as engineering, economics, medical science and social science. At present, work on the soft set theory is progressing rapidly. P.K.Maji et al. [2] defined basic properties of soft set theory. Aktaş and Çağman [3] compared to soft sets to the related concepts of fuzzy sets and rough sets and introduced soft group and derived their basic properties. Afterward, soft algebraic structures have been studied by some researchers, such as soft ring, soft field and soft modules [5], soft int-groups [4]. Soft linear spaces and soft norm on soft linear spaces are given and some of their properties are studied by Samanta, Das ve P. Majumdar [7]. In [8] Q. Sun, Z. Zang and J. Liu, introduced the definition of soft modules and constructed some basic properties of soft modules, Many important results could be proved only for representations over algebraically closed fields. Module theoretic approach is better suited to deal with deeper results in representation theory. This is the subject matter of representation theory [9, 10, 11]. Soon after the introduction of fuzzy set theory by L.A. Zadeh [12] in 1965, Rosenfeld [13] initiated the fuzzification of algebraic structures. Recently, some researchers studied G-modules on fuzzy sets.

As a continuation of these works S. Fernandez [14] introduced fuzzy parallels of the notions of G-modules, group representations, reducibility, irreducibility and completely reducibility and observe, some of their basic properties. In [15] A.K.Sinro and K. Dewangan studied isomorphism theorems for fuzzy submodules of G-modules. Recently, many authors have studied some algebraic structures of soft set theory. [16, 17, 18, 19, 20] Some interesting results in the theory of soft modules are still being explored currently. However the theory of soft modules has not yet been studied. M.Shabir [21] gave some new notions such as the restricted intersection, the restricted union, the restricted difference and the extended intersection of two soft sets along with a new notion of complement of a soft set. The work of this paper is organized as follows. In the second section as preliminaries, we give basic concepts of soft sets and fuzzy soft G-modules. In Section 3, we introduce IFSG-modules and study their characteristic properties. In Section 4, we give the applications of IFSG-modules.
2. PRELIMINARIES

In this section as a beginning, the concepts of G-module [22] soft sets introduced by Molodsov [1] and the notions of fuzzy soft set introduced by Maji et al. [23] have been presented.

2.1 Definition (Molodtsov [4]): Let U be an initial universe, P(U) be the power set of U, E be the set of all parameters and A ⊆ E. A soft set (f_A, E) on the universe U is defined by the set of order pairs

\[ (f_A(e), e) \in P(U) \]  where \( f_A : E \rightarrow P(U) \) such that \( f_A(e) = \emptyset \) if \( e \notin A \).

Here \( f_A \) is called an approximate function of the soft set.

Example: Let U = \{u_1, u_2, u_3, u_4\} be a set of four shirts and E = \{ yellow(p_1), green(p_2), black(p_3) \} be a set of parameters.

If A = \{p_1, p_2\} ⊆ E, \( f_A(p_1) = \{u_1, u_2, u_3, u_4\} \) and \( f_A(p_2) = \{u_1, u_2, u_3\} \), then we write the soft set \( (f_A, E) = \{(p_1, u_1, u_2, u_3, u_4), (p_2, u_1, u_2, u_3)\} \) over U which describe the “colour of the shirts” which Mr. X is going to buy.

2.2 Definition (P.K.Maji [23]): Let U be an initial universe, E be the set of all parameters and A ⊆ E. A pair (F, A) is called a fuzzy soft set over U where F: A → P(U) is a mapping from A into P(U), where P(U) denotes the collection of all fuzzy subsets of U.

Example: Consider the above example, here we cannot express with only two real numbers 0 and 1, we can characterize it by a membership function instead of crisp number 0 and 1, which associate with each element a real number in the interval [0, 1]. Then \( (f_A, E) = \{(p_1, \{u_1, 0.7\}, \{u_2, 0.5\}, \{u_3, 0.4\}, \{u_4, 0.2\}), (p_2, \{u_1, 0.5\}, \{u_2, 0.1\}, \{u_3, 0.5\}) \} \) is the fuzzy soft set representing the “colour of the shirts” which Mr. X is going to buy.

2.3 Definition (Curties [9]): Fuzzy soft class, Let U be an initial Universe set and E be the set of attributes. Then the pair (U, E) denotes the collection of all fuzzy soft sets on U with attributes from E and is called a fuzzy soft class.

Definition 2.4 (Ali et al. [21]): Let F_A and G_B be two soft sets over U such that A ⊆ E. A mapping \( \phi : M → M^* \) is a G-module homomorphism if

1. \( \phi \) (k_1 m_1 + k_2 m_2) = k_1 \phi (m_1) + k_2 \phi (m_2)
2. \( \phi \) (gm) = g \phi (m), \( k_1, k_2 \in K, m, m_1, m_2 \in M \) & \( g \in G \).

2.6 Definition (A’gman al et al. [22]): Let F_A and G_B be be soft sets over the common universe U and \( \psi \) be a function from A to B. Then we can define the soft set \( \psi(F_A) \) over U, where \( \psi(F_A) : B → P(U) \) is a set valued function defined by

\[ \psi(F_A)(b) = {_{A \in A} F_A(a) | a \in A} \] and \( \psi(a) = b \).

If \( \psi^{-1}(b) \neq \emptyset \), \( \phi \) = 0 otherwise for all b ∈ B. Here, \( \psi(F_A) \) is called the soft image of \( F_A \) under \( \psi \). Moreover we can define a soft set \( \psi^{-1}(G_B) \) over U, where \( \psi^{-1}(G_B) : A → P(U) \) is a set valued function defined by \( \psi^{-1}(G_B)(a) = G(\psi(a)) \) for all a ∈ A. Then, \( \psi^{-1}(G_B) \) is called the soft pre image (or inverse image) of \( G_B \) under \( \psi \).

2.1. Theorem (A’gman al et al. [22]): Let \( F_A \) and \( T_K \) be soft sets over U, \( F^r_A \), \( T^r_K \) be their relative soft sets, respectively and \( \psi \) be a function from H to K. Then,

i) \( \psi^{-1}(T^r_K) = (\psi^{-1}(T_K))^r \)
ii) \( \psi(F^r_A) = (\psi(F_A))^r \) and \( \psi^{-1}(F^r_A) = (\psi^{-1}(F_A))^r \).

3. IFSG-MODULES

In this section, we first define intersection fuzzy soft G-modules of a vector space, abbreviated as IFSG-modules.

We then investigate its related properties with respect to soft set operations.

Let G be a non-empty set. A fuzzy subset \( \mu \) on G is defined by \( \mu : G → [0, 1] \) for all \( x ∈ G \).

3.1. Definition: Let G be a group. Let M be a G-module of V and A_M be a fuzzy soft set over V. Then A_M is called Intersection Fuzzy Soft G-module of V (IFSG-m), denoted by A_M ⊆ V if the following properties are satisfied

\( (IFSG-m_1) \) A(ax + by) ≥ A(x) ∩ A(y)
\( (IFSG-m_2) \) A(ax) ≥ A(αx) for all x, y ∈ M, a, b, α ∈ F.
Example: Let \( G = \{1, -1\}, M = R^4 \) over \( R \). Then \( M \) is a G-module.

Define \( A \) on \( M \) by,

\[
A(x) = \begin{cases} 
1 & \text{if } x_i = 0 \forall i, \\
0.5 & \text{if } \text{atleast } x_i \neq 0.
\end{cases}
\]

Where \( x = \{x_1, x_2, x_3, x_4\} \); \( x_i \in R \). Then \( A \) is a fuzzy soft G-Module.

3.1. Proposition: If \( A_M <_1 V \), then \( A(0v) \supseteq A(x) \) for all \( x \in M \).

Proof: Since \( A_M \) is an IFSG-module of \( V \), then \( A(ax+by) \supseteq A(x) \cap A(y) \) for all \( x, y \in M \) and since \( (M, +) \) is a group, if we take \( ay = -ax \) then, for all \( x \in M \), \( A(ax-ax) = A(0v) \supseteq A(x) \cap A(y) = A(x) \).

3.2. Proposition: If \( A_{M_1} <_i V \) and \( B_{M_2} <_i V \), then \( A_{M_1} \cap B_{M_2} <_i V \).

Proof: Since \( M_1 \) and \( M_2 \) are G-modules of \( V \), then \( M_1 \cap M_2 \) is a G-module of \( V \). By definition 2.6, let \( A_{M_1} \cap B_{M_2} = (A, M_1) \cap (B, M_2) = (T, M_1 \cap M_2) \),

Where, \( T(x) = A(x) \cap B(x) \) for all \( x \in M_1 \cap M_2 \neq \emptyset \). Then for all \( x, y \in M_1 \cap M_2 \) and \( \alpha \in F \).

\[
\begin{align*}
(\text{IFSG-m}_1) & \quad T(ax+by) = (A(ax+by)) \cap (B(ax+by)) \supseteq (A(x) \cap A(y)) \cap (B(x) \cap B(y)) \\
& = (A(x) \cap B(x)) \cap (A(y) \cap B(y)) = T(x) \cap T(y),
\end{align*}
\]

(\text{IFSG-m}_2) \quad T(ax) = A(ax) \cap B(ax) \supseteq A(x) \cap B(x) = T(x).

\[
\text{There for } A_{M_1} \cap B_{M_2} = T_{M_1 \cap M_2} <_i V.
\]

3.2. Definition: Let \( (A, M_1) \) and \( (B, M_2) \) be two IFSG-modules of \( V_1 \) and \( V_2 \) respectively, the product of IFSG-modules \( (A, M_1) \) and \( (B, M_2) \) is defined as \( (A, M_1) \times (B, M_2) = (Q, M_1 \times M_2) \), where \( Q(x,y) = A(x) \times B(y) \) for all \( (x, y) \in M_1 \times M_2 \).

3.1. Theorem: If \( A_{M_1} <_i V \) and \( B_{M_2} <_i V \), then \( A_{M_1} \times B_{M_2} <_i V \).

Proof: Since \( M_1 \) and \( M_2 \) are G-modules of \( V_1 \) and \( V_2 \) respectively, then \( M_1 \times M_2 \) is a G-module of \( V_1 \times V_2 \). By definition 3.2, let \( A_{M_1} \times B_{M_2} = (A, M_1) \times (B, M_2) = (Q, M_1 \times M_2) \),

where \( Q(x, y) = A(x) \times B(y) \) for all \( (x, y) \in M_1 \times M_2 \).

Then for all \( (x_1, y_1), (x_2, y_2) \in M_1 \times M_2 \) and \( (\alpha_1, \alpha_2) \in F \times F \),

\[
\begin{align*}
(\text{IFSG-m}_1) & \quad Q((ax_1, by_1) + (ax_2, by_2)) = Q(ax_1 + ax_2, by_1 + by_2) \\
& = A(ax_1 + ax_2) \times B(by_1 + by_2) \\
& \supseteq (A(ax_1) \cap A(ax_2)) \times (B(by_1) \cap B(by_2)) \\
& = Q(x_1, y_1) \cap Q(x_2, y_2)
\end{align*}
\]

(\text{IFSG-m}_2) \quad Q((\alpha_1, \alpha_2)(x_1, y_1)) = Q(\alpha_1 x_1 + \alpha_2 y_1) \\
= A(\alpha_1 x_1 + \alpha_2 y_1) \supseteq A(x_1) \cap B(y_1) = Q(x_1, y_1).

Hence \( A_{M_1} \times B_{M_2} = Q_{M_1 \times M_2} <_i V \times V \).

3.3. Definition: Let \( A_{M_1} \) and \( B_{M_2} \) be two IFSG-module’s of \( V \). If \( M_1 \cap M_2 = \{0_v\} \), then the sum of IFSG-module’s \( A_{M_1} \)and \( B_{M_2} \) is defined as \( A_{M_1} + B_{M_2} = T_{M_1 + M_2} \) where \( T(ax+by) = A(x)+B(y) \) for all \( ax+by \in M_1 + M_2 \).

3.2. Theorem: If \( A_{M_1} <_i V \) and \( B_{M_2} <_i V \) where \( M_1 \cap M_2 = \{0_v\} \), then \( A_{M_1} + B_{M_2} <_i V \).

Proof: Since \( M_1 \& M_2 \) are G-modules of \( V \), then \( M_1 \times M_2 \) is a G-modules of \( V \). By definition: 3.3,

Let \( A_{M_1} + B_{M_2} = (A, M_1) + (B, M_2) = (T, M_1 + M_2) \), where \( T(ax+by) = A(x)+B(y) \) for all \( ax+by \in M_1 + M_2 \). It is obvious that since \( M_1 \cap M_2 = \{0_v\} \), then the sum is well defined. Then for all \( ax_1 + by_1, ax_2 + by_2 \in M_1 + M_2 \) and \( \alpha \in F \),

© 2016, RJPA. All Rights Reserved 350
3.4. Definition: Let $A_M$ be an IFSG-module of $V$. Then,
   
   (i) $A_M$ is said to be trivial if $A(x) = \{0_V\}$ for all $x \in M$.
   
   (ii) $A_M$ is said to be whole if $A(x) = V$ for all $x \in M$.

3.3. Proposition: Let $AM_1$ and $BM_2$ be two IFSG-modules of $V$, then
   
   (i) If $AM_1$ and $BM_2$ are trivial IFSG-modules of $V$, then $AM_1 \cap BM_2$ is a trivial IFSG-module of $V$.

   (ii) If $AM_1$ and $BM_2$ are whole IFSG-modules of $V$, then $AM_1 \cap BM_2$ is a whole IFSG-module of $V$.

   (iii) If $AM_1$ is a trivial IFSG-module of $V$ and $BM_2$ is a whole IFSG-modules of $V$, then $AM_1 \cap BM_2$ is a trivial IFSG-module of $V$.

   (iv) If $AM_1$ and $BM_2$ are trivial IFSG-modules of $V$ where $M_1 \cap M_2 = \{0_V\}$, then $AM_1 + BM_2$ is a trivial IFSG-module of $V$.

   (v) If $AM_1$ and $BM_2$ are whole IFSG-modules of $V$ where $M_1 \cap M_2 = \{0_V\}$, then $AM_1 + BM_2$ is a whole IFSG-module of $V$.

Proof: The proof is easily seen by definition 2.4, definition 3.3, definition 3.4 and theorem 3.1.

3.4. Proposition: Let $AM_1$ and $BM_2$ be two IFSG-modules of $V_1$ and $V_2$ respectively. Then
   
   (i) If $AM_1$ and $BM_2$ are trivial IFSG-modules of $V_1$ and $V_2$ respectively, then $AM_1 \times BM_2$ is a trivial IFSG-module of $V_1 \times V_2$.

   (ii) If $AM_1$ and $BM_2$ are whole IFSG-modules of $V_1$ and $V_2$ respectively, then $AM_1 \times BM_2$ is a whole IFSG-module of $V_1 \times V_2$.

Proof: The proof is easily seen by definition 3.2 and definition 3.4

Applications of IFSG modules: In this section, we give the applications of soft image, soft pre image, upper $\alpha$-inclusion of fuzzy soft sets and linear transformation of vector spaces with respect to IFSG-modules.

4.1. Theorem: If $AM \subset V$, then $M_G = \{x \in M / A(x) = A(0_V)\}$ is a $G$-module of $M$.

Proof: It is obvious that $0_V \in M_G$ and $0_G \neq M_G \subseteq M$. We need to show that $ax+by \in M_G$ and $ax \in M_G$ for all $x, y \in M_G$ and $a \in F$, which means that $A(ax+by) = A(0_V)$ and $A(ax) = A(0_V)$ have to be satisfied. Since $x, y \in M_G$ and $AM$ is an IFSG-Module of $V$, then $A(x) = A(y) = A(0_V)$, $A(ax+by) \supseteq A(x) \cap A(y) = A(0_V)$, $A(ax) \supseteq A(x) = A(0_V)$ for all $x, y \in M_G$ and $a \in F$. Moreover, by Proposition 3.1, $A(0_V) \supseteq A(ax+by)$ and $A(0_V) \supseteq A(ax)$ which completes the proof.

4.2. Theorem: Let $AM$ be a fuzzy soft set over $V$ and $\alpha$ be a subset of $V$ such that $A(0_V) \supseteq \alpha$. If $AM$ is an IFSG-module of $V$, then $AM^{2n}$ is a $G$-module of $V$.

Proof: Since $A(0_V) \supseteq \alpha$, then $0_V \in AM^{2n}$ and $0_V \notin AM^{2n} \subseteq V$. Let $x, y \in AM^{2n}$, then $A(x) \supseteq \alpha$ and $A(y) \supseteq \alpha$. We need to show that $x + y \in AM^{2n}$ for all $x, y \in AM^{2n}$. Since $AM$ is an IFSG-Module of $V$, it follows that $A(ax+by) \supseteq A(x) \cap A(y) \supseteq \alpha \cap \alpha = \alpha$. Furthermore, $A(nx) \supseteq A(x) \supseteq \alpha$, which completes the proof.

4.3. Theorem: Let $AM$ and $T_W$ be fuzzy soft sets over $V$, where $M$ and $W$ are $G$-modules of $\gamma$ and $\Psi$ be a linear isomorphism from $M$ to $W$. If $AM$ is an IFSG-Module of $V$, then so is $\Psi(AM)$.
**Proof:** Let \( w_1, w_2 \in W \). Since \( \Psi \) is a subjective linear transformation. Then there exists \( m_1, m_2 \in M \) such that \( \Psi (m_1) = w_1, \Psi (m_2) = w_2 \). Then

\[
(\Psi (A_M))(aw_1 + bw_2) = \bigcup \{ A(m) : m \in M, \Psi(m) = aw_1 + bw_2 \}
= \bigcup \{ A(m) : m \in M, m = \Psi^{-1}(aw_1 + bw_2) \}
= \bigcup \{ A(m) : m \in M, m = \Psi^{-1}(w_1 + w_2) = am_1 + bm_2 \}
= \bigcup \{ A(am_1 + bm_2) : m_1 \in M, \Psi(m_1) = w_1 + w_2 \} = \bigcup \{ A(m_1) \cap A(m_2) : m_1, m_2 \in M, \Psi(m_1) = w_1, \Psi(m_2) = w_2 \}
\]

Now let \( \alpha \in F \) and \( w \in W \). Since \( \Psi \) is a surjective linear transformation, there exists \( m \in M \) such that \( \Psi(m) = w \). Then

\[
(\Psi (A_M))(aw) = \bigcup \{ A(m) : m \in M, \Psi(m) = aw \}
= \bigcup \{ A(m) : m \in M, m = \Psi^{-1}(aw) \}
= \bigcup \{ A(m) : m \in M, m = \Psi^{-1}(\Psi(aw)) = \Psi(aw) \}
= \bigcup \{ A(\alpha m) : \alpha m \in M, \Psi(\alpha m) = w \}
= (\Psi (A_M))(w)
\]

Hence, \( \Psi (A_M) \) is an IFSG –module of \( V \).

**4.4. Theorem:** Let \( A_M \) and \( TW \) be fuzzy soft sets over \( V \), where \( M \) and \( W \) are \( G \)-modules of \( \gamma \) and \( \Psi \) be a linear isomorphism from \( M \) to \( W \). If \( T_W \) is an IFSG-Module of \( V \), then so is \( \Psi^{-1} (T_W) \).

**Proof:** Let \( m_1, m_2 \in M \). Then

\[
\Psi^{-1} (T_W)(am_1 + bm_2) = T(\Psi(am_1 + bm_2)) = T(\Psi(am_1) + \Psi(bm_2)) \supseteq T(\Psi(m_1)) \cap T(\Psi(m_2)) = (\Psi^{-1} (T_W))(m_1) \cap (\Psi^{-1} (T_W))(m_2)
\]

Now let \( \alpha \in F \) and \( m \in M \). Then,

\[
\Psi^{-1} (T_W)(\alpha m) = T(\alpha \Psi(m)) \supseteq T(\Psi(m)) = \Psi^{-1} (T_W)(m)
\]

Hence \( \Psi^{-1} (T_W) \) is an IFSG –module of \( V \).

**4.5. Theorem:** Let \( V_1 \) and \( V_2 \) be two vector spaces and \( (A_1, M_1) \preceq_i V_1, (A_2, M_2) \preceq_i V_2 \). If \( f : M_1 \to M_2 \) is a linear transformation of vector spaces, then

(i) \( f \) is surjective, then \( (A_1, f^{-1}(M_2)) \preceq_i V_1 \),
(ii) \( (A_2, f(M_1)) \preceq_i V_2 \),
(iii) \( (A_1, \ker f) \preceq_i V_1 \).

**Proof:**

(i) Since \( M_1 < V_1 \), \( M_2 < V_2 \) and \( f : M_1 \to M_2 \) is a surjective linear transformation, then it is clear that \( f^{-1}(M_2) < V_1 \). Since \( (A_1, M_1) \preceq_i V_1 \) and \( f^{-1}(M_2) < M_1 \), \( A_1(ax + by) \supseteq A(x) \cap A(y) \) and \( A_1(ax) \supseteq A(x) \) for all \( x, y \in f^{-1}(M_2) \) and \( \alpha \in F \). Hence \( (A_1, f^{-1}(M_2)) \preceq_i V_1 \).
(ii) Since \( M_1 < V_1 \), \( M_2 < V_2 \) and \( f : M_1 \to M_2 \) is a vector space linear transformation, then \( f(M_1) < V_2 \).
Since \( f(M_1) \subseteq M_2 \), the result is obvious by definition 3.1.
(iii) Since \( \ker f < V_1 \) and \( \ker f \subseteq M_1 \), the rest of the proof is clear by definition 3.1.

**4.1. Corollary:** Let \( (A_1, M_1) \preceq_i V_1 \), \( (A_2, M_2) \preceq_i V_2 \). If \( f : M_1 \to M_2 \) is a linear transformation, then \( (A_2, \{0M_2\}) \preceq_i V_2 \).

**Proof:** By theorem: 4.5, (iii) \( (A_1, \ker f) \preceq_i V_1 \), then \( (A_2, f(\ker f)) = (A_2, \{0M_2\}) \preceq_i V_2 \), By theorem 4.5 (ii).

**CONCLUSION**

Throughout this paper, we have dealt with IFSG-modules of a vector space. We have investigated their related properties with respect to soft set operations. Furthermore, we have derived some applications of IFSG-modules with respect to soft image, soft pre image, soft anti image, Further study could be done for fuzzy soft sub structures of different algebras.
REFERENCES

12. L. A. Zadeh, Fuzzy sets, Information and Control, 8 (1965) 338 - 353

Source of Support: Nil, Conflict of interest: None Declared

[Copy right © 2016, RJPA. All Rights Reserved. This is an Open Access article distributed under the terms of the International Research Journal of Pure Algebra (IRJPA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]