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ON PRODUCT OF RANGE QUATERNION HERMITIAN MATRICES

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ABSTRACT

In this paper we discuss the product of q-EP matrices are discussed.

Keywords: *Moore-Penrose inverse, q-EP matrix, product of q-EP.*

INTRODUCTION

Through we shall deal with nxn quaternion matrices [7]. Let A* denote the conjugate transpose of A. Let A^- be the generalized inverse of A satisfying AA^-A and z be the Moore-Penrose of A[6]. Any matrix $A \in H_{nXn}$ is called q-EP (2) if R(A)=R(A^*) and his called q-EP_r, if A is q-EP and rk(A)=r, where N(A), R(A) and rk(A) denote the null space, range space and rank of A respectively. It is well known that sum and sum of parallel summable q-EP matrices are q-EP [3]. In general the product of symmetric, Hermitian, normal and EP respectively. Similarly, the product of q-EP matrices need not be q-EP. For instance

Let A = $\begin{pmatrix} 1 & 1+i+j+k \\ 1-i-j-k & 2 \end{pmatrix}$

B = $\begin{pmatrix} 3 & 1+2i+3j+4k \\ 1-2i-3j-4k & 4 \end{pmatrix}$

A is q-EP and B is q-EP.

AB = $\begin{pmatrix} 13-4j-2k & 5+6i+7j+4k \\ 5-7i-9j-11k & 18+2i+4k \end{pmatrix}$ is not q- EP

Theorem 1.1: Let A_1 and A_n (n>a) be q-EP_r matrices and let A = $A_1A_2A_3\ldots\ldots A_n$. Then the following statements are equivalent:

(i) A is q-EP_r
(ii) R(A_1) = R(A_n) and rk(A)=r
(iii) R(A_1^*) = R(A_n^*) and rk(A) = r

Proof:
(i) ⇔ (ii): Since A_1 and A_n are q-EP_r, therefore R(A_1) = R(A_1^*) and R(A_n) = R(A_n^*). Let A=$A_1A_2A_3\ldots\ldots A_n$.

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Since $A_1, A_2, A_3, \ldots\ldots A_n$ are q-EP
$\Rightarrow A = A_1A_2A_3\ldots\ldots A_n$

$R(A) \subseteq R(A_1)$ and $rk(A) = rk(A_1)$
$\Rightarrow R(A) = R(A_1)$.

Also $A^* = (A_n^*)(A_{n-1}^*)\ldots\ldots(A_1^*)$
$\Rightarrow R(A^*) \subseteq R(A_n^*)$ and $rk(A) = rk(A_n) = r$
$\Rightarrow rk(A^*) = rk(A_n^*) = r$

Therefore,
$R(A^*) = R(A_n^*)$

Now,
A is q-EP_r $\Leftrightarrow R(A) = R(A^*)$ and $rk(A) = r$ (By definition q-EP[2])
$\Leftrightarrow R(A_1) = R(A_n^*)$
$\Leftrightarrow R(A_n^*) = R(A_n)$
$\Leftrightarrow R(A_1) = R(A_n)$ and $rk(A) = r$

(ii) $\Leftrightarrow$ (iii):
$R(A_1) = R(A_n)$
$\Leftrightarrow R(A_1^*) = R(A_n)^* = R(A_n^*)$
$\Leftrightarrow R(A_1^*) = R(A_n^*)$
Hence the theorem

Corollary 1.2: Let A and B are q-EP_r matrices. Then AB is q-EP_r $\Leftrightarrow$ rk(AB) = r and R(A) = R(B)

Proof: Proof follows from theorem (1.1) for the product of two q-EPr matrices A and B.

Remarks 1.3: In the corollary both the conditions that rk(AB) = r and R(A) = R(B) are essential for the product of two q-EPr matrices to be q-EPr. This can be seen in the following example.

Example 1.4:

Let $A = \begin{pmatrix} 1 & k \\ -k & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -k \\ k & 0 \end{pmatrix} \Rightarrow AB = \begin{pmatrix} -2 & -k \\ k & -1 \end{pmatrix}$

A is q-EP and B is q-EP., then AB is q-EP $\Leftrightarrow$ rk(AB) = 2 and R(A) = R(B)

Example 1.5:

Let $A = \begin{pmatrix} 1 & 1+i+j+k \\ 1-i-j-k & 2 \end{pmatrix}$

$B = \begin{pmatrix} 3 & 1+2i+3j+4k \\ 1-2i-3j-4k & 4 \end{pmatrix}$

A is q-EP and B is q-EP. $R(A) \neq R(B)$. Then

$AB = \begin{pmatrix} 1 - 43j - 2k & 5+6i+7j+5k \\ 5-7i-9j-11k & 18+2i+4k \end{pmatrix}$ is not q-EP

AB is not q-EP and rk(AB) = 2

Theorem 1.6: Let $rk(AB) = rk(B) = r_1$ and $rk(BA) = rk(A) = r_2$. If AB, B are q-EP_{r1} and A is q-EP_{r2} then BA is q-EP_{r2}

Proof: Since $rk(BA) = rk(A) = r_2$, It is enough to show that $N(BA) = N((BA)^*)$ to prove BA is q-EP_{r2}

Now, $N(A) \subseteq N(BA)$ and $rk(BA) = rk(A)$
$\Rightarrow N(A) = N(BA)$

Also, $N(B) \subseteq N(AB)$ and $rk(AB) = rk(B)$
$\Rightarrow N(B) = N(AB)$

Now N(BA) $= N(A)$
$= N(A^*)$
$\subseteq N(B^*A^*)$
$= N(AB)$
$= N(B)$
$= N(B^*)$
$\subseteq N(A^*B^*)$
$= N(BA)^*)$
N(BA) s $\subseteq N(BA)^*)$

Further $rk(BA) = rk(BA)^*$
$\Rightarrow N(BA) = N((BA)^*)$

Thus, BA is q-EP_{r2}

Hence the theorem.

Lemma 1.7: A, B $\in H_{nxn}$ be of rank r.
(i) $rk(AA^*) = rk(A^*A)$
(ii) $rk(AB) = rk(B) - \dim \left[N(A) - N(B^*)^* \right]$
If A and B are q-EP_r matrices and AB has rank r, then BA has rank r.

Proof: By theorem [1], $rk(AB) = rk(B) - \dim(N(A) \cap N(B^*)^{\perp}$

Since $rk(AB) = rk(B) = r$
$N(A) \cap N(B^*)^{\perp} = \{0\} \Leftrightarrow N(A) \cap N(B)^{\perp} = \{0\}$. [Since B is q-$EP_r$]
$\Rightarrow N(A)^{\perp} \cap N(B) = \{0\}$
$\Rightarrow N(A^*)^{\perp} \cap N(B) = \{0\}$ [Since A is q-EP_r]

Now, $rk(BA) = rk(B)(A)$
$= rk(A) - \dim (N(B) \cap N(A^*)^{\perp})$
$= rk(A) - 0$
$= rk(A)$

That is $rk(BA) = r$

Hence the lemma.

Example 1.8:

$$A = \begin{pmatrix} 1 & i+j \\ -i-j & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix}$$

A and B are q-EP_r matrices
$\therefore$ rk(A) = r, rk(B) = r

$$AB = \begin{pmatrix} j-i & k \\ 0 & j-i \end{pmatrix}$$

$\therefore$ rk(AB) = r

$$\text{Then } BA = \begin{pmatrix} -j+1 & 0 \\ -k & -j+i \end{pmatrix}$$

rk(BA) = r

Theorem 1.9: If A, B and AB are q-EP_r matrices then BA is q-EP_r.

Proof: Since A, B are q-EPr matrices and rk(AB) = r, by lemma(1.7), rk(BA) = r. Now the theorem follows from theorem (1.6) for $r_1=r_2=r$.

Hence the theorem.

Example 1.10:

$$A = \begin{pmatrix} 0 & k & j \\ -k & 0 & 0 \\ -j & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -k & -j \\ k & 0 & 0 \\ j & 0 & 0 \end{pmatrix}$$

A and B are q-EP_r Matrices

$$AB = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & -i \\ 0 & i & -1 \end{pmatrix}$$

And AB is q-EP_r matrices

$$BA = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & -i \\ 0 & i & -1 \end{pmatrix}$$

So, if A, B and AB are Q-EP matrices then BA is q-EP_r

Corollary 1.9: Let A, B be q-EP_r matrices. Then the following statements are equivalent

(i) AB is qEP_r

(ii) $(AB)^{\dagger}$ is q-EP_r

(iii) $A^{\dagger}B^{\dagger}$ is q-EP_r

(iv) $B^{\dagger}A^{\dagger}$ is q-EP_r

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