Research Journal of Pure Algebra -1(7), 2011, Page: 152-160
RJPA Available online through www.rjpa.info ISSN 2248–9037

NUMERICAL SOLUTION OF FUZZY DIFFERENTIAL EQUATIONS BY RUNGE-KUTTA METHOD OF ORDER TWO

Dr. C. Duraisamy and B. Usha*

Department of Mathematics, Kongu Engineering College, Perundurai, Erode-638 052, Tamilnadu, India
E-mail address: vikash_usha@yahoo.co.in

(Received on: 15-09-11; Accepted on: 06-10-11)

ABSTRACT

In this paper, we have introduced and studied a new technique for getting the solution of fuzzy initial value problem.

Mathematics Subject Classification: *34A12; 65L05.*

Keywords: *Fuzzy differential equations, Runge-Kutta method of order two, Parallelogram fuzzy number.*

1. INTRODUCTION

The study of fuzzy differential equations (FDEs) forms a suitable setting for model dynamical systems in which uncertainties or vagueness pervade. First order linear and non-linear FDEs are one of the simplest FDEs which appear in many applications. In the recent years, the topic of FDEs has been investigated extensively. The organized of the paper is as follows. In the first three sections below, we recall some concepts and introductory materials to deal with the fuzzy initial value problem. In section five, we present Runge-Kutta method of order two and its iterative solution for solving of Fuzzy differential equations. The proposed algorithm is illustrated by an example in the last section.

2. PRELIMINARY

A parallelogram fuzzy number u is defined by four real numbers $k < l < m < n$, where the base of the parallelogram is the interval $[k, n]$ and its vertices at $x = l, x = m$. Parallelogram fuzzy number will be written as $u=(k, \ell, m, n)$.The membership function for the parallelogram fuzzy number $u = (k, \ell, m, n)$ is defined as the following:

$$u(x) = \begin{cases} \dfrac{x-k}{l-k}, & k \le x \le l \\ 1, & l \le x \le m \\ \dfrac{x-n}{m-n}, & m \le x \le n \end{cases} \tag{1}$$

we will have :

$u > 0$ if $k > 0$; $u > 0$ if $l > 0$; $u > 0$ if $m > 0$; & $u > 0$ if $n > 0$.

Let us denote R_F by the class of all fuzzy subsets of R (i.e. $u : R \to [0,1]$) satisfying the following properties:

(i) $\forall\ u \in R_F$, u is normal, i.e. $\exists\ x_0 \in R$ with $u(x_0) = 1$;

(ii) $\forall\ u \in R_F$, u is convex fuzzy set (i.e. $u(tx + (1 - t) y) \ge \min\{u(x), u(y)\}$, $\forall\ t \in [0,1]$, $x, y \in R$);

(iii) $\forall\ u \in R_F$, u is upper semi continuous on R;

(iv) $\{\overline{x \in R; u(x) > 0}\}$ is compact, where $\overline{A}$ denotes the closure of A.

Corresponding author: B. Usha, *E-mail: vikash_usha@yahoo.co.in

Then R_F is called the space of fuzzy numbers. Obviously $R \subset R_F$. Here $R \subset R_F$ is understood as $R = \{\chi_{\{x\}}; x \text{ is usual real number}\}$. We define the *r*-level set, $x \in R$;

$$[u]_r = \{x \setminus u(x) \geq r\}, \qquad 0 \leq r \leq 1; \tag{2}$$

Clearly, $[u]_0 = \{x \setminus u(x) > 0\}$ is compact,

which is a closed bounded interval and we denote by $[u]_r = [\underline{u}(r), \overline{u}(r)]$. It is clear that the following statements are true,

1. $\underline{u}(r)$ is a bounded left continuous non decreasing function over [0,1],
2. $\overline{u}(r)$ is a bounded right continuous non increasing function over [0,1],
3. $\underline{u}(r) \leq \overline{u}(r)$ for all $r \in (0,1]$,

for more details see [2],[3].

Let $D: R_F \times R_F \to R_+ \cup \{0\}$,

$D(u,v) = \text{Sup}_{r\in[0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$ be Hausdorff distance between fuzzy numbers, where $[u]_r = [\underline{u}(r), \overline{u}(r)]$, $[v]_r = [\underline{v}(r), \overline{v}(r)]$. The following properties are well-known:

$D(u + w, v + w) = D(u, v), \ \forall\ u,v,w \in R_F,$

$D(k.u, k.v) = |k| D(u, v), \ \forall\ k \in R,\ u,v \in R_F,$

$D(u + v, w + e) \leq D(u, w) + D(v, e), \ \forall\ u, v, w, e \in R_F$ and (R_F, D) is a complete metric space.

Lemma: 2.1 If the sequence of non-negative numbers $\{W_n\}_{n=0}^{N}$ satisfy $|W_{n+1}| \leq A|W_n| + B$, $0 \leq n \leq N-1$, for the given positive constants A and B, then

$$|W_n| \leq A^n |W_0| + B\frac{A^n - 1}{A - 1}, \qquad 0 \leq n \leq N.$$

Lemma: 2.2 If the sequence of numbers $\{W_n\}_{n=0}^{N}$, $\{V_n\}_{n=0}^{N}$ satisfy

$$|W_{n+1}| \leq |W_n| + A \max\{|W_n|, |V_n|\} + B,$$

$$|V_{n+1}| \leq |V_n| + A \max\{|W_n|, |V_n|\} + B,$$

for the given positive constants A and B, then denoting

$$U_n = |W_n| + |V_n|, \qquad 0 \leq n \leq N,$$

we have,

$$U_n \leq \overline{A}^n U_0 + \overline{B}\frac{\overline{A}^n - 1}{\overline{A} - 1}, \ 0 \leq n \leq N,$$

where $\overline{A} = 1 + 2A$ and $\overline{B} = 2B$.

3. FUZZY INITIAL VALUE PROBLEM

Consider a first-order fuzzy initial value differential equation is given by

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{3}$$

where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$ is a parallelogram or a parallelogram shaped fuzzy number.

We denote the fuzzy function y by $y = [\underline{y}, \overline{y}]$. It means that the r-level set of y(t) for $t \in [t_0, T]$ is

$$[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)], \quad [y(t_0)]_r = [\underline{y}(t_0;r), \overline{y}(t_0;r)], \; r \in (0,1]$$

we write $f(t, y) = [\underline{f}(t, y), \overline{f}(t, y)]$ and

$$\underline{f}(t, y) = F[t, \underline{y}, \overline{y}], \quad \overline{f}(t, y) = G[t, \underline{y}, \overline{y}].$$

Because of $y' = f(t, y)$ we have

$$\underline{f}(t, y(t); r) = F[t, \underline{y}(t;r), \overline{y}(t;r)] \tag{4}$$

$$\overline{f}(t, y(t); r) = G[t, \underline{y}(t;r), \overline{y}(t;r)] \tag{5}$$

By using the extension principle, we have the membership function

$$f(t, y(t))(s) = \sup \{y(t)(\tau) \backslash s = f(t,\tau)\}, \; s \in R \tag{6}$$

so fuzzy number $f(t, y(t))$. From this it follows that

$$[f(t, y(t))]_r = [\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)], \; r \in (0,1], \tag{7}$$

where
$$\underline{f}(t, y(t); r) = \min \{f(t, u) \mid u \in [y(t)]_r\} \tag{8}$$

$$\overline{f}(t, y(t); r) = \max \{f(t, u) \mid u \in [y(t)]_r\}. \tag{9}$$

Definition: 3.1 A function f: $R \rightarrow R_F$ is said to be fuzzy continuous function, if for an arbitrary fixed $t_0 \in R$ and $\in > 0$, $\delta > 0$ such that

$$|t - t_o| < \delta \Rightarrow D[f(t), f(t_0)] < \in \quad \text{exists, [14].}$$

Throughout this paper we also consider fuzzy functions which are continuous in metric D. Then the continuity of $f(t, y(t); r)$ guarantees the existence of the definition of $f(t, y(t); r)$ for $t \in [t_o, T]$ and $r \in [0,1]$ [8]. Therefore, the functions G and F can be definite too.

4. RUNGE - KUTTA METHOD OF ORDER TWO

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{10}$$

It is known that, the sufficient conditions for the existence of a unique solution to (10) are that f to be continuous function satisfying the Lipschitz condition of the following form:

$$\| f(t,x) - f(t,y) \| \le L \| x - y \|, \; L > 0.$$

We replace the interval $[t_0, T]$ by a set of discrete equally spaced grid points, $t_0 < t_1 < t_2 < \ldots\ldots\ldots < t_N = T$, $h = \frac{T - t_0}{N}, t_i = t_0 + ih, i = 0,1,\ldots\ldots, N$ to obtain the Euler's method for the system (10).

By the mean value theorem of integral calculus, we obtain

$$y(t_{n+1}) = y(t_n) + hf(t_n + \theta h, y(t_n + \theta h)) + O(h^3), \quad 0 < \theta < 1 \tag{11}$$

If we approximate $y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n))$ in the argument of *f*, we get

$$y(t_{n+1}) = y(t_n) + hf(t_{n+1}, y(t_n)) + hf(t_n, y(t_n)) + O(h^3). \quad (12)$$

If we set

$$K_1 = hf(t_n, y(t_n))$$
$$K_2 = hf(t_{n+1}, y(t_n) + K_1)$$

We get the method as $y(t_{n+1}) = y(t_n) + K_2 + O(h^3)$. (13)

when $\theta = 1/2$, we obtain

$$y(t_{n+1}) = y(t_n) + hf(t_n + h/2, y(t_n + h/2)) + O(h^3). \quad (14)$$

However, $t_n + (h/2)$ is not a nodal point.

If we approximate $y(t_n + h/2)$ in (14) by Euler's method with spacing *h/2*, we get,

$$y(t_n + h/2) = y(t_n) + h/2 f(t_n, y(t_n)) + O(h^3).$$

Then, we have the approximation

$$y(t_{n+1}) = y(t_n) + hf(t_n + h/2, y(t_n) + h/2 f(t_n, y(t_n))) + O(h^3). \quad (15)$$

If we set

$$K_1 = hf(t_n, y(t_n))$$
$$K_2 = hf(t_n + h/2, y(t_n) + K_1/2)$$

then (15) can be written as, $y(t_{n+1}) = y(t_n) + K_2$ as $h \to 0$. (16)

Alternately, if we use the approximation

$$y'(t_n + h/2) = 1/2\,(y'(t_n) + y'(t_n + h))$$

and the Euler method , we obtain

$$y'(t_n + h/2) = 1/2(f(t_n, y(t_n)) + f(t_{n+1}, y(t_n) + hf(t_n, y(t_n))))$$

Thus (14) may be approximated by

$$y(t_{n+1}) = y(t_n) + h/2(f(t_n, y(t_n)) + f(t_{n+1}, y(t_n) + hf(t_n, y(t_n)))) \quad (17)$$

If we set

$$K_1 = hf(t_n, y(t_n))$$
$$K_2 = hf(t_{n+1}, \underline{y}(t_n) + K_1)$$

then (17) can be written as,

$$y(t_{n+1}) = y(t_n) + 1/2(K_1 + K_2) \quad (18)$$

The proposed algorithm is illustrated by an example in last section.

5. RUNGE- KUTTA METHOD OF ORDER TWO FOR SOLVING FUZZY DIFFERENTIAL EQUATIONS

Let $Y = [\underline{Y}, \overline{Y}]$ be the exact solution and $y = [\underline{y}, \overline{y}]$ be the approximated solution of the fuzzy initial value problem (3).

Let $[Y(t)]_r = [\underline{Y}(t;r), \overline{Y}(t;r)]$, $[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)]$.

Throughout this argument, the value of r is fixed. Then the exact and approximated solution at t_n are respectively denoted by

$[Y(t_n)]_r = [\underline{Y}(t_n;r), \overline{Y}(t_n;r)]$, $[y(t_n)]_r = [\underline{y}(t_n;r), \overline{y}(t_n;r)]$ $(0 \le n \le N)$.

The grid points at which the solution is calculated are

$$h = \frac{T - t_0}{N}, t_i = t_0 + ih, \quad 0 \le i \le N$$

Then we obtain, $\underline{Y}(t_{n+1};r) = \underline{Y}(t_n;r) + \frac{1}{2}[K_1 + K_2]$

where

$$\begin{aligned} K_1 &= hF\left[t_n, \underline{Y}(t_n;r), \overline{Y}(t_n;r)\right] \\ K_2 &= hF\left[t_{n+1}, \underline{Y}(t_n;r), \overline{Y}(t_n;r) + K_1\right] \end{aligned} \tag{19}$$

and $\overline{Y}(t_{n+1};r) = \overline{Y}(t_n;r) + \frac{1}{2}[K_1 + K_2]$

where

$$\begin{aligned} K_1 &= hG\left[t_n, \underline{Y}(t_n;r), \overline{Y}(t_n;r)\right] \\ K_2 &= hG\left[t_{n+1}, \underline{Y}(t_n;r), \overline{Y}(t_n;r) + K_1\right] \end{aligned} \tag{20}$$

And we have $\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) + \frac{1}{2}[K_1 + K_2]$

where

$$\begin{aligned} K_1 &= hF\left[t_n, \underline{y}(t_n;r), \overline{y}(t_n;r)\right] \\ K_2 &= hF\left[t_{n+1}, \underline{y}(t_n;r), \overline{y}(t_n;r) + K_1\right] \end{aligned} \tag{21}$$

$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) + \frac{1}{2}[K_1 + K_2]$$

]
and

where

$$\begin{aligned} K_1 &= hG\left[t_n, \underline{y}(t_n;r), \overline{y}(t_n;r)\right] \\ K_2 &= hG\left[t_{n+1}, \underline{y}(t_n;r), \overline{y}(t_n;r) + K_1\right] \end{aligned} \tag{22}$$

Clearly, $\underline{y}(t;r)$ and $\overline{y}(t;r)$ converge to $\underline{Y}(t;r)$ and $\overline{Y}(t;r)$, respectively whenever $h \to 0$ [4].

6. NUMERICAL RESULTS

In this section, the exact solutions and approximated solutions are obtained by Runge-kutta methods of order two are plotted in Figure 1 and Figure 2.

Example: 6.1

Consider the initial value problem [11]

$$\begin{cases} y'(t) = tf(t), & t \in [0,1] \\ y(0) = \left(1.01 + 0.1\, r\sqrt{e}\ ,\ 1.5 + 0.1\, r\sqrt{e}\right). \end{cases}$$

The exact solution at $t = 0.1$ is given by

$$Y(0.1;\mathrm{r}) = \left[\left(1.01 + 0.1\, r\sqrt{e}\right)e^{0.005}, \left(1.5 + 0.1\, r\sqrt{e}\right)e^{0.005}\right], \quad 0 \le \mathrm{r} \le 1.$$

Using iterative solution of Runge-Kutta method, we have

$$\underline{y}(0;r) = 1.01 + 0.1\, r\sqrt{e}, \quad \overline{y}(0;r) = 1.5 + 0.1\, r\sqrt{e}$$

and by

$$\underline{y}^{(0)}(t_{i+1};r) = \underline{y}(t_i;r) + h\,\underline{y}(t_i;r)$$

$$\overline{y}^{(0)}(t_{i+1};r) = \overline{y}(t_i;r) + h\,\overline{y}(t_i;r),$$

where $i = 0, 1, \ldots, N-1$ and $h = \dfrac{1}{N}$. Now, using these equations as an initial guess for following iterative solutions respectively,

$$\underline{y}^{j}(t_{i+1};r) = \underline{y}(t_i;r) + \frac{1}{2}[K_1 + K_2],$$

Where

$$K_1 = h\,\underline{y}(t_i;r)$$

$$K_2 = h\left[\underline{y}(t_i;r) + K_1\right]$$

and

$$\overline{y}^{j}(t_{i+1};r) = \overline{y}(t_i;r) + \frac{1}{2}[K_1 + K_2],$$

where

$$K_1 = h\,\overline{y}(t_i;r)$$

$$K_2 = h\left[\overline{y}(t_i;r) + K_1\right]$$

and $j = 1, 2, 3$. Thus, we have $\underline{y}(t_i;r) = \underline{y}^{(3)}(t_i;r)$ and $\overline{y}(t_i;r) = \overline{y}^{(3)}(t_i;r)$, for $i = 1, \ldots, N$. Therefore, $\underline{Y}(1;r) \approx \underline{y}^{(3)}(1;r)$ and $\overline{Y}(1;r) \approx \overline{y}^{(3)}(1;r)$ are obtained.

Table 1: Exact solution

r	Exact solution
0	2.174625 , 2.582368
0.1	2.208604 , 2.616346
0.2	2.242583 , 2.650325
0.3	2.276561 , 2.684303
0.4	2.310540 , 2.718282
0.5	2.344518 , 2.752260
0.6	2.378497 , 2.786239
0.7	2.412475 , 2.820217
0.8	2.446454 , 2.854196
0.9	2.480432 , 2.888174
1	2.514411 , 2.922153

Table 2: Approximated solution

h / *r*	0.1	0.01	0.001
0	1.116578 , 1.658284	1.025314 , 1.522744	1.017142 , 1.510606
0.1	1.134805 , 1.676511	1.042052 , 1.539481	1.033745 , 1.527210
0.2	1.153032 , 1.694738	1.058789 , 1.556218	1.050349 , 1.543814
0.3	1.171259 , 1.712965	1.075526 , 1.572956	1.066953 , 1.560418
0.4	1.189486 , 1.731192	1.092263 , 1.589693	1.083557 , 1.577022
0.5	1.207713 , 1.749419	1.109000 , 1.606430	1.100161 , 1.593625
0.6	1.225940 , 1.767646	1.125738 , 1.623167	1.116764 , 1.610229
0.7	1.244167 , 1.785873	1.142475 , 1.639904	1.133368 , 1.626833
0.8	1.262394 , 1.804100	1.159212 , 1.656642	1.149972 , 1.643437
0.9	1.280621 , 1.822327	1.175949 , 1.673379	1.166576 , 1.660041
1	1.298848 , 1.840554	1.192686 , 1.690116	1.183180 , 1.676644

Table 3: Error for different values of *r* and h.

h / *r*	0.1	0.01	0.001
0	0.252280	0.025477	0.005167
0.1	0.255595	0.025812	0.005235
0.2	0.258909	0.026146	0.005302
0.3	0.262223	0.026481	0.005370
0.4	0.265538	0.026816	0.005438
0.5	0.268852	0.027151	0.005506
0.6	0.272166	0.027485	0.005574
0.7	0.275480	0.027820	0.005642
0.8	0.278794	0.028155	0.005710
0.9	0.282109	0.028429	0.005778
1	0.285423	0.028824	0.005845

Graphical Representation of exact and approximated solution

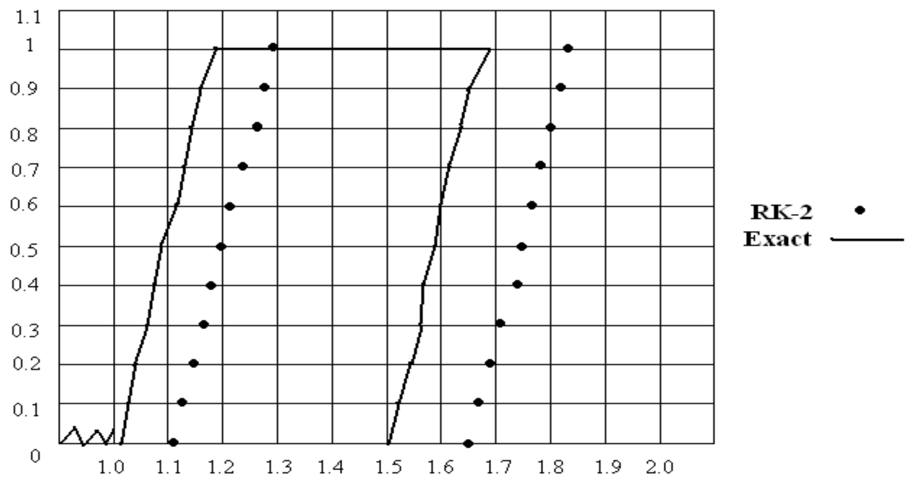

Figure 1 : h = 0.1

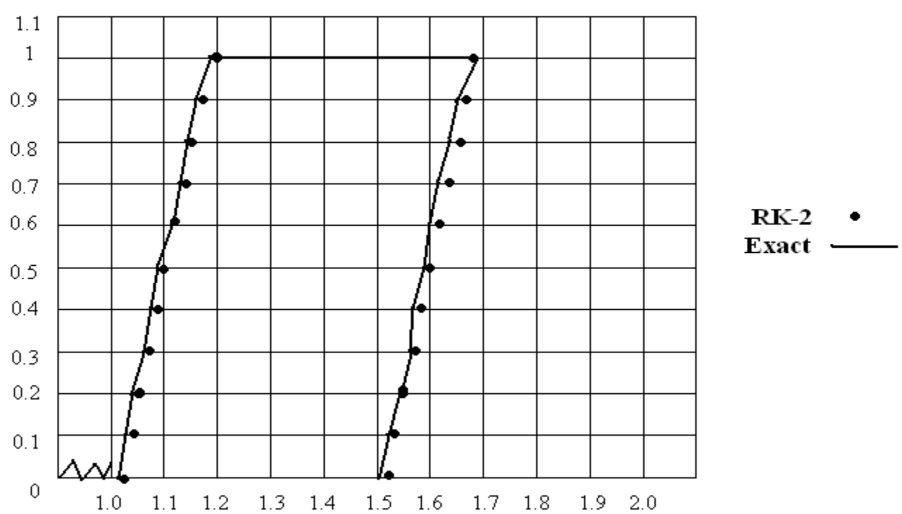

Figure 2 : h = 0.01

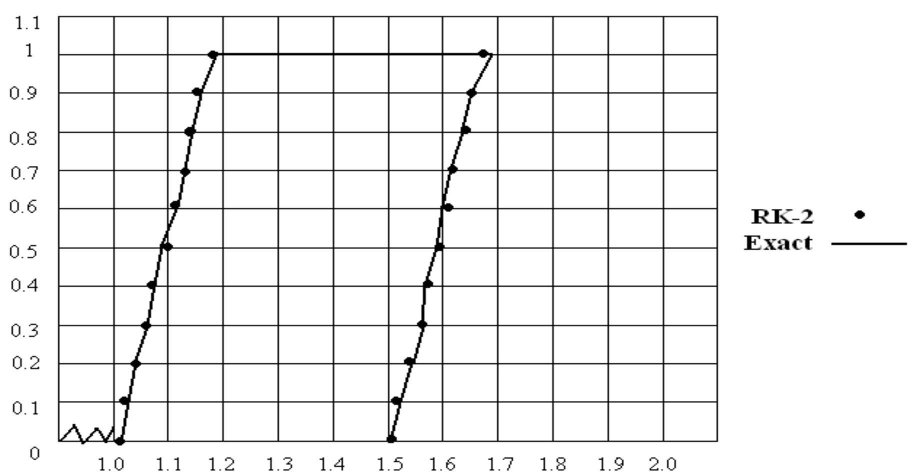

Figure 3 : h = 0.001

REFERENCES:

[1] Abbasbandy .S and Allahviranloo .T, Numerical solutions of fuzzy differential equations by Taylor method, *Journal of Computational Methods in Applied Mathematics.* Vol.2. 113 -124, 2002.

[2] Buckley .J. J and Eslami .E, Introduction to Fuzzy Logic and Fuzzy Sets, *Physica- Verlag, Heidelberg, Germany.* 2001.

[3] Buckley .J. J and Eslami .E and Feuring .T, Fuzzy Mathematics in Economics and Engineering, *Physica-Verlag, Heidelberg, Germany*. 2002.

[4] Duraisamy .C and Usha .B, Another Approach to Solution of Fuzzy Differential Equations, *Applied Mathematical Sciences* Vol.4, 2010, No.16, 777-790.

[5] Duraisamy .C and Usha .B, Another Approach to Solution of Fuzzy Differential Equations by Modified Euler's Method, International conference on communication and computational Intelligence, published in IEEE explore.

[6] Duraisamy .C and Usha .B, Numerical Solution of Fuzzy Differential Equations by Runge-kutta Method, National conference on scientific computing and Applied Mathematics.

[7] Duraisamy .C and Usha .B, Numerical Solution of Fuzzy Differential Equations by Taylor method, *International journal of Mathematical Archive* Vol.2, 2011.

[8]Duraisamy .C and Usha .B, Solution of Fuzzy Differential Equations, *Bulletin of Pure and Applied Mathematics,* Accepted for publication.

[9] Duraisamy .C and Usha .B, Solving Fuzzy Differential Equations by Modified Euler's Method, *The Journal of The Indian Academy of Mathematics,* Accepted for publication.

[10] Duraisamy .C and Usha. B, Solving Fuzzy Differential Equations by Runge-Kutta Method of order four, National Conference on Discrete and Fuzzy Mathematics.

[11] Duraisamy .C and Usha .B, Solving Fuzzy Differential Equations by Runge- kutta Method of order two , National conference on Advances in Mathematical Analysis & Application.

[12] Goetschel .R and Voxman. W, Elementary Calculus, *Fuzzy Sets and Systems,* 18 (1986) 31 - 43.

[13] Kaleva .O, Fuzzy differential equations, *Fuzzy Sets and Systems*, 24 (1987), pp. 301 – 317.

****

