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NILPOTENCY OF IDEALS GENERATED BY SETS CONTAINED IN THE CENTER

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ABSTRACT

In this paper we consider R be a nonassociative and noncommutative ring. Let S be an additive subgroup of R such that (S, R) = 0. Now we take V={x∈R/ (x, y) = 0, for all y∈R}. From (S, R)=0, it follows that s∈V, where s is in S, and V is subring of R. Using these we show that V equals the center C of R, the set I=S+SR is an ideal of R and $(S+SR)^n = S^n+S^nR$ for all positive integers n. Also it is proved that the ideal of R generated by S is nilpotent if and only if the subring generated by S is nilpotent.

***Mathematics subject classification**: Primary 17A30.*

***Keywords:** Associator, Commutator, Nucleus, Center, Nilpotent ideal.*

INTRODUCTION

Yen and Hentzel [3] studied the nonassociative rings with the ideal generated by sets contained in two of the three nuclei. In this paper we consider R be a nonassociative and noncommutative ring. The associator (a, b, c) and commutator (x, y) are defined by (a, b, c) = (ab)c–a(bc), (x, y)=xy-yx for all a, b, c, x, y in R. The nucleus N and center C of R are defined by N={n∈R/(n, R, R}= (R, n, R)=(R, R, n=0} and center C={c∈N/(c, R)=0}. The ideal of R is nilpotent if there is a positive integer n such that every product involving n elements is zero. In any nonassociative ring we have the Teichmuller identity (ab, c, d)-(a, bc, d)+(a, b, cd)=a(b, c, d)+(a, b, c)d. Thus (R, R, R) + (R, R, R)R=(R, R, R)+R(R, R, R). Kleinfeld [1] showed that (R, R, R)+(R, R, R)R is an ideal of R. This is called the associator ideal. It is the ideal which is generated by all associators. Similarly, we have (R, R)+(R, R)R=(R, R)+R(R, R). Let S be an additive subgroup of R such that (S, R) = 0. So S+SR=S+RS. Examples of S are (R, R) and (R, R, R). Now we take V={x∈R/ (x, y) = 0, for all y∈R}. From (S, R) = 0, it follows that s∈V where s is in S and V is a subring of R. Using these we show that V equals the center C of R. Thus S is contained in the center C of R. Then we prove that the set I=S+SR is an ideal of R and $(S+SR)^n=S^n+S^nR$ for all positive integers n. Also we show that the ideal of R generated by S is nilpotent if and only if the subring generated by S is nilpotent.

PRELIMINARIES

Let R be a nonassociative and noncommutative ring. Let S be an additive subgroup of R such that

$$(S, R)=0 \tag{1}$$

Now we take V={x∈R/(x, y) = 0, for all y∈R}. From (1) it follows that s∈V where s is in S and V is a subring of R. We now prove the following lemmas.

Lemma 1: The set W= {s/s∈V, Rs⊂V} is an ideal of R such that (x, y, s)∈W and (s, y, x)∈W, for s∈V and all x, y∈R.

Proof: From (1), we see that W is a two sided ideal of R. From the Teichmuller identity
a(b, c, d)+(a, b, c)d = (ab, c, d)-(a, bc, d)+(a, b, cd), which holds in any ring, we get z(x, y, s)=(zx, y, s) -(z, xy, s) +(z, x, ys)-(z, x, y)s∈V, since V is a subring of R and (1) holds. Similarly we get z(s, y, x) = (s, y, x)z∈V. Hence (x, y, s)∈W and (s, y, x)∈W.

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Lemma 2: Let R be a ring without non zero ideals $\neq$ R satisfying (S, R) = 0. Then V equals the center C of R.

Proof: The ideal W of lemma 1 is contained in V of R. Since R has no non trivial ideal either W=R or W = 0. If W = R, then R is commutative, which is a contradiction to our assumption. Hence W $\neq$ R. So W=0. Then from lemma1 we get (x, y, s) = 0 and (s, y, x) = 0, for all $s \in S$, $x, y \in R$. We know the semijacobi identity (x, z, y)=(x, y, z)+(z, x, y)-(xy, z) +(x, z)y-x(y, z), which holds in any ring, we get (x, s, y) = (x, y, s)+(s, x, y)-(xy, s)+(x, s)y-x(y, s) = 0, from (1), (x, y, s) = 0 and (s, y, x)=0. Hence S contained in the nucleus N of R. Therefore V equals the center C of R.

MAIN RESULTS

From lemma 2 we have that S is contained in the center C of R. Let the set I=S+SR. From (1) we have S+SR=S+RS. By assumption $SR \subset I$ and $Rs \subset I$.

Lemma 3: If S is an additive subgroup of R such that (S, R) = 0, then the set I=S+SR is a two sided ideal of R.

Proof: Since S is in the center of R, RI=R(S+SR)=RS+R.SR=RS+RS. $R \subset I+IR$ and IR=(S+RS)R=SR+RS.R=SR+R.SR $\subset$ I+RI. Hence I+IR=I+RI. Now IR=(S+SR)R=SR+SR.R =SR+$SR^2 \subset$I. So I is a right ideal of R. Since I+IR=I+RI, we have that I is a left ideal of R. Hence I is an ideal of R.

Lemma 4: If S is an additive subgroup of R such that (S, R)=0, then $S^n+S^nR=S^n+RS^n$ for all positive integers n.

Proof: $\sum_{i=1}^{\infty} S^i$ is an associative subring contained in the center of R. So $(S^i, S^j, R) = (S^i, R, S^j) = (R, S^i, S^j) = 0$ for all integers i, j $\geq$ 1. By induction, it is true for n=1. We assume the result for n. Then we get $S^{n+1}R=S^nS.R=S^n.SR \subset S^n(S+RS)=S^{n+1}+S^n.RS=S^{n+1}+S^nR.S \subset S^{n+1}+(S^n+S^nR)S=S^{n+1}+ S^nR.S =S^{n+1}+RS^n.S=S^{n+1}+R.S^{n+1}$ and $RS^{n+1}=R.S^nS=RS^n.S \subset (S^n+S^nR)S =S^{n+1}+ S^nR.S=S^{n+1} +S^n.RS \subset S^{n+1} +S^n.(S+SR) = S^{n+1}+S^n.SR=S^{n+1}+S^{n+1}.R$.

So, $S^{n+1}+S^{n+1}R=S^{n+1}+RS^{n+1}$, for all positive integers n. This proves the lemma.

Lemma 5: If S is an additive subgroup of R such that (S, R) = 0, then $S^n+S^nR=S^n+RS^n$ is the ideal of R generated by S^n for all positive integers n.

Proof: From lemma 4, we have $S^n+S^nR=S^n+RS^n$. If we replace S by S^n in lemma 3, we get the ideal $S^n+S^nR=S^n+RS^n$, where n is any positive integer.

Lemma 6: If S is an additive subgroup of R such that (S, R) = 0, then $S^iR.S^jR \subset S^{i+j}+S^{i+j}R$ for all integers i, j $\geq$ 1.

Proof: Since S is in the center C of R, by lemma 5, $S^iR.S^jR=S^i.R(S^jR) \subset S^i.(S^jR) \subset S^i(S^j+S^jR)=S^{i+j}+S^i(S^jR)=S^{i+j}+(S^i.S^j).R=S^{i+j}+S^{i+j}R$.

Lemma 7: If S is an additive subgroup of R such that (S, R) = 0, then $S^iR.S^j \subset S^{i+j}+S^{i+j}R$ for all integers i, j $\geq$ 1.

Proof: Since S is in the center C of R and using lemma 5, $S^iR.S^j=S^i.RS^j \subset S^i(S^j+S^jR) =S^{i+j}+S^i.S^jR = S^{i+j}+S^{i+j}.R$.

Lemma 8: If S is an additive subgroup of R such that (S, R) = 0, then $(S+SR)^n = S^n+S^nR$ for all positive integers n.

Proof: We assume the result for all positive integers m $\leq$ n. Then using this inductive hypothesis, the lemma 7 and lemma 6 for all integers i and j, I $\leq$n and j$\leq$n, we get $(S+SR)^{i+j}=(S+SR)^i (S+SR)^j=(S^i+S^iR) (S^j+S^jR) =S^{i+j}+S^i.S^jR+ S^iR.S^j+S^iR.S^jR = S^{i+j}+S^iS^j.R+S^iR.S^j+S^iR.S^jR=S^{i+j}+S^{i+j}R$.

Therefore $(S+SR)^{i+j} = S^{i+j}+S^{i+j}R$.

This proves the lemma

Now we prove the following theorem.

Theorem: Let R be a non associative, non commutative ring and S be an additive subgroup of R such that (S, R) = 0. The ideal of R generated by S is nilpotent if and only if the subring generated by S in nilpotent.

Proof: If the ideal of R generated by S is nilpotent then $(S+SR)^n = 0$. From lemma 8 it follows that $S^n+S^nR = 0$. So the subring generated by S is nilpotent. The converse follows similarly.

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