THE PRODUCT SPAN OF SUM SPAN OF A SUBSET OF A COMPLETELY BOUNDED ARTEX SPACE OVER A BI-MONOID

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ABSTRACT

When Sum Combination is introduced, it was introduced only for a finite subset of an Artex Space A over a bi-monoid M. Now the sum span for any subset of a completely bounded Artex space over a bi-monoid is defined. When Product Combination is introduced, it was introduced only for a finite subset of an Artex Space over a bi-monoid. Now the Product span for any subset of a completely bounded Artex space over a bi-monoid is defined. The product span of sum span of a subset of a completely bounded artex space over a bi-monoid is defined. Propositions were found and proved.

Keywords: Bi-monoids, Artex Spaces over bi-monoids, Completely Bounded Artex Spaces over bi-monoids, Sum Combination, Sum Span, Product Combination, Product Span, Product Span of Sum Span.

I. INTRODUCTION

The algebraic system Bi-semi-group is more general to the algebraic system ring or an associative ring. Artex Spaces over Bi-monoids were introduced. As a development of Artex Spaces over Bi-monoids, SubArtex spaces of Artex spaces over bi-monoids were introduced. From the definition of a SubArtex space, it is clear that not every subset of an Artex space over a bi-monoid is a SubArtex space. Some propositions which qualify subsets to become SubArtex Spaces were found and proved. Completely Bounded Artex Spaces over bi-monoids were introduced. It contains the least and greatest elements namely 0 and 1. When Sum Combination was introduced, it was introduced only for a finite subset of an Artex Space A over a bi-monoid M. Now the sum span for any subset of a completely bounded Artex space over a bi-monoid is defined. When Product Combination was introduced, it was introduced only for a finite subset of an Artex Space over a bi-monoid. Now the Product span for any subset of a completely bounded Artex space over a bi-monoid is defined. Now sum combination, sum span, product combination and product span together give a new SubArtex Space namely product span of sum span of a subset of a completely bounded Artex space over a bi-monoid. It will be useful for the development of the theory of Artex Spaces over bi-monoids

II. PRELIMINARIES

2.1. Semi-group: A non-empty set S together with a binary operation. is called a Semi-group if for all a, b, c ∈ S, a.(b . c) = (a.b).c

2.2. Monoid: A non-empty set N together with a binary operation , is called a monoid if
  (i) (i) for all a, b, c ∈ N, a.(b . c) = (a.b).c and
  (ii) there exists an element denoted by e in N such that a.e = a = e.a, for all a ∈ N.

The element e is called the identity element of the monoid N.

2.3. Relation: Let S be a non-empty set. Any subset of S×S is called a relation in S.

If R is a relation in S, then R is a subset of S×S.

If (a,b) belongs to the relation R, then we can express this by aRb or by a ≤ b.

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Note: A relation may be denoted by ≤

2.4. Partial Ordering: A relation ≤ on a set P is called a partial order relation or a partial ordering in P if
   (i) a ≤ a, for all a ϵ P  ie ≤ is reflexive,
   (ii) a ≤ b and b ≤ a implies a = b  ie ≤ is anti-symmetric, and
   (iii) a ≤ b and b ≤ c implies a ≤ c  ie ≤ is transitive.

2.5. Partially Ordered Set (POSET): If ≤ is a partial ordering in P, then the ordered pair (P, ≤) is called a Partially Ordered Set or simply a POSET.

2.6. Lattice: A lattice is a partially ordered set (L, ≤) in which every pair of elements a, b ϵ L has a greatest lower bound and a least upper bound.

The greatest lower bound of a and b is denoted by aΛb and the least upper bound of a and b is denoted by a∨b

2.7. Lattice as an Algebraic System: A lattice is an algebraic system (L, Λ, V) with two binary operations Λ and V on L which are both commutative, associative and satisfy the absorption laws namely Λ(aVb) = a and V(aΛb) = a, for all a, b ϵ L

The operations Λ and V are called cap and cup respectively, or sometimes meet and join respectively.

2.8. Properties: We have the following properties in a lattice (L, Λ, V )
   1. a Λ a = a  1’.a V a = a (Idempotent Law)
   2. a Λ b = b Λ a  2’.a V b = b V a (Commutative Law)
   3. (a Λ b) Λ c = a Λ (b Λ c)  3’.(a V b) V c = a V (b V c) (Associative Law)
   4. a Λ (a V b) = a  4’.a V (a Λ b) = a, for all a, b, c ϵ L (Absorption Law)

2.9. Complete Lattice: A lattice is called a complete lattice if each of its nonempty subsets has a least upper bound and a greatest lower bound.

Every finite lattice is a complete lattice and every complete lattice must have a least element and a greatest element.

The least and the greatest elements, if they exist, are called the bounds or units of the lattice and are denoted by 0 and 1 respectively.

2.10. Bounded Lattice: A lattice which has both elements 0 and 1 is called a bounded lattice. A bounded lattice is denoted by (L, Λ, V, 0, 1)

The bounds 0 and 1 of a lattice (L, Λ, V) satisfy the following identities.
   For any a ϵ L,   a V 0 = a    a Λ 1 = a    a V 1 = 1    a Λ 0 = 0

2.10.1. Example: For any set S, the lattice (P(S), ⊆) is a bounded lattice. Here for each A, B ϵ P(S), the least upper bound of A and B is A∪B and the greatest lower bound of A and B is A∩B. The bounds in this lattice are φ, the empty set and S, the universal set.

2.11. Complemented Lattice: Let (L, Λ, V, 0, 1) be a bounded lattice. An element a’ϵL is called a complement of an element a ϵ L if a a’ = 0, a V a’ = 1. A bounded lattice (L, Λ, V, 0, 1) is said to be a complemented lattice if every element of L has at least one complement. A complemented lattice is denoted by (L, Λ, V,’, 0, 1).

2.11.1. Example: For any set S, the lattice (P(S), ⊆) is a Complemented lattice.

For each A, B ϵ P(S), the least upper bound of A and B is A∪B and the greatest lower bound of A and B is A∩B.

The bounds in this lattice are φ, the empty set and S, the universal set.

Here for any A ϵ P(S), the complement of A in P(S) is S-A

2.12. Doubly Closed Space: A non-empty set D together with two binary operations denoted by + and . is called a Doubly Closed Space if (i) a.(b+c) = a.b + a.c and (ii) (a+b).c = a.c + b.c, for all a, b, c ϵ D

A Doubly closed space is denoted by (D, +, . )
Note-1: The axioms (i) a.(b+c) = a.b + a.c and (ii) (a+b).c = a.c + b.c, for all a, b, c ∈ D are called the distributive properties of the Doubly Closed Space.

Note-2: The operations + and . need not be the usual addition and usual multiplication respectively.

2.12.1. Example: Let N be the set of all natural numbers.

Then (N, +, .), where + is the usual addition and . is the usual multiplication, is a Doubly closed space.

Similarly (Z, +, .), (Q, +, .), (R, +, .) and (C, +, .) are all Doubly closed spaces.

2.12.2. Example: (Z, +, −), where + is the usual addition and − is the usual subtraction, is not a Doubly closed space.

2.13. Bi-semi-group: A Doubly closed space (S, +, .) is called a Bi-semi-group if + and . are associative in D.

2.13.1. Example 2.2.1: (N, +, .), (Z, +, .), (Q, +, .), (R, +, .), and (C, +, .), where + is the usual addition and . is the usual multiplication, are all Bi-semi-groups.

2.14. Bi-monoid: A Bi-semi-group (M, +, .) is called a Bi-monoid if there exist elements denoted by 0 and 1 in S such that a+0=a=0+a, for all a∈M and a.1=a=1.a, for all a∈M.

The element 0 is called the identity element of M with respect to the binary operation + and the element 1 is called the identity element of M with respect to the binary operation.

2.14.1. Example: Let W = {0, 1, 2, 3,…}.Then (W, +, .), where + is the usual addition and . is the usual multiplication, is a Bi-monoid.

2.14.2. Example: Let Q’=Q^+∪{0}, where Q^+ is the set of all positive rational numbers. Then (Q’, +, .) is a bi-monoid.

2.14.3. Example: R’=R^+∪{0}, where R^+ is the set of all positive real numbers. Then (R’, +, .) is a bi-monoid.

2.14.4. Example: (Z, +, .), (Q, +, .), (R, +, .), and (C, +, .), where + is the usual addition and . is the usual multiplication, are all Bi-monoids.

2.15. Artex Space Over a Bi-monoid: Let (M, +, .) be a bi-monoid with the identity elements 0 and 1 with respect to + and . respectively. A non-empty set A together with two binary operations ^ and v is said to be an Artex Space Over the Bi-monoid (M, +, .) if

1. (A, ^, v) is a lattice and
2. for each m∈M, m0, and a∈A, there exists an element ma∈A satisfying the following conditions:
   (i) m(a ^ b) = ma ^ mb
   (ii) m(a v b) = ma v mb
   (iii) ma ^ na ≤ (m + n)a and ma v na ≤ (m + n)a
   (iv) (mn)a = m(na), for all m, n∈M, m0, m0, and a, b∈A
   (v) 1.a = a, for all a∈A.

Here, ≤ is the partial order relation corresponding to the lattice (A, ^, v).The multiplication ma is called a bi-monoid multiplication with an artex element or simply bi-monoid multiplication in A.

2.15.1. Example: Let W = {0, 1, 2, 3,…}.

Then (W, +, .) is a bi-monoid, where + and . are the usual addition and multiplication respectively.

Let Z be the set of all integers

Then (Z, ≤) is a lattice in which ^ and v are defined by a ^ b = minimum of {a, b} and a v b = maximum of {a, b}, for all a, b ∈ Z.

Clearly for each m∈W, m0, and for each a∈Z, ma∈Z.

Also,

(i) m(a ^ b) = ma ^ mb
(ii) m(a v b) = ma v mb
(iii) ma ^ na ≤ (m + n)a and ma v na ≤ (m + n)a
(iv) (mn)a = m(na)
(v) 1.a = a, for all m, n∈W, m + 0, n + 0 and a, b∈Z.
Therefore, Z is an Artex Space Over the Bi-monoid (W, +, .)

2.15.2. Example: As defined in Example 2.15.1, Q, the set of all rational numbers is an Artex space over W.

2.15.3. Example: As defined in Example 2.15.1, R, the set of all real numbers is an Artex space over W.

2.15.4. Example: Let Q' = Q \cup \{0\}, where Q' is the set of all positive rational numbers.

Then (Q', +, .) is a bi-monoid. Now as defined in Example 2.15.1, Q, the set of all rational numbers is an Artex space over Q'.

2.15.5. Example: R' = R \cup \{0\}, where R' is the set of all positive real numbers. Then (R', +, .) is a bi-monoid.

As defined in Example 2.15.1, R, the set of all real numbers is an Artex space over R'.

2.16. Properties

2.16.1. Properties: We have the following properties in a lattice (L, \Lambda, V)

1. a \Lambda a = a
2. a \Lambda b = b \Lambda a
3. (a \Lambda b) \Lambda c = a \Lambda (b \Lambda c)
4. a \Lambda (a \Lambda b) = a

1'. a V a = a
2'. a V b = b V a
3'. (a V b) V c = a V (b V c)
4'. a V (a V b) = a

Therefore, we have the following properties in an Artex Space A over a bi-monoid M.

(i) m(a \Lambda a) = ma
(ii) m(a \Lambda b) = m(b \Lambda a)
(iii) m(a \Lambda b) \Lambda c = m(a \Lambda (b \Lambda c))
(iv) m(a \Lambda (a \Lambda b)) = ma

(i)' m(a V a) = ma
(ii)' m(a V b) = m(b V a)
(iii)' m(a V b) V c = m(a V(b V c))
(iv)' m(a V (a V b)) = ma

2.17. SubArtex Space: Let (A, \Lambda, V) be an Artex space over a bi-monoid. (M, +, .). Let S be a nonempty subset of A. Then S is said to be a SubArtex Space of A if (S, \Lambda, V) itself is an Artex Space over M.

2.17.1. Example: As defined in Example 2.15.1, Z is an Artex Space over W = \{0, 1, 2, 3, .....\} and W is a subset of Z. Also W itself is an Artex space over W under the operations defined in Z. Therefore, W is a SubArtex space of Z.

2.18. Complete Artex Space over a bi-monoid: An Artex space A over a bi-monoid M is said to be a Complete Artex Space over M if as a lattice, A is a complete lattice that is each nonempty subset of A has a least upper bound and a greatest lower bound.

2.18.1. Remark: Every Complete Artex space must have a least element and a greatest element.

The least and the greatest elements, if they exist, are called the bounds or units of the Artex space and are denoted by 0 and 1 respectively.

2.19. Lower Bounded Artex Space over a bi-monoid: An Artex space A over a bi-monoid M is said to be a Lower Bounded Artex Space over M if as a lattice, A has the least element 0.

2.20. Upper Bounded Artex Space over a bi-monoid: An Artex space A over a bi-monoid M is said to be an Upper Bounded Artex Space over M if as a lattice, A has the greatest element 1.

2.21. Bounded Artex Space over a bi-monoid: An Artex space A over a bi-monoid M is said to be a Bounded Artex Space over M if A is both a Lower bounded Artex Space over M and an Upper bounded Artex Space over M.

2.22. Completely Bounded Artex Space over a bi-monoid: A Bounded Artex Space A over a bi-monoid M is said to be a Completely Bounded Artex Space over M if (i) 0.a = 0, for all a \in A (ii) m.0 = 0, for all m \in M.

2.22.1. Note: While the least and the greatest elements of the Complemented Artex Space is denoted by 0 and 1, the identity elements of the bi-monoid (M, +, .) with respect to addition and multiplication are, if no confusion arises, also denoted by 0 and 1 respectively.
III. THE SUM SPAN OF A SUBSET OF AN ARTEX SPACE OVER A BI-MONOID

3.1. Sum Combination: Let \((A, \Lambda, V)\) be an Artex Space over a bi-monoid \((M, +, .)\). Let \(a_1, a_2, a_3, \ldots, a_n \in A\). Then any element of the form \(m_1a_1Vm_2a_2Vm_3a_3V \ldots \ V m_na_n\), where \(m_i \in M\), is called a Sum Combination or Join Combination of \(a_1, a_2, a_3, \ldots. a_n\).

3.2. The Sum Span of a subset of an Artex Space over a Bi-monoid: Let \((A, \Lambda, V)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, .)\) and \(W\) be a nonempty finite subset of \(A\). Then the Sum Span of \(W\) or Join Span of \(W\) denoted by \(S[W]\) is defined to be the set of all sum combinations of elements of \(W\). That is, if \(W = \{a_1, a_2, a_3, \ldots. a_n\}\), then \(S[W] = \{m_1a_1Vm_2a_2Vm_3a_3V \ldots \ V m_na_n / m_i \in M\}\).

3.3. PROPOSITIONS

Proposition 3.3.1: Let \((A, \Lambda, V)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, .)\) and \(W\) be a nonempty finite subset of \(A\). Then \(W \subseteq S[W]\)

Proposition 3.3.2: Let \((A, \Lambda, V)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, .)\). Let \(W\) and \(V\) be any two nonempty finite subsets of \(A\). Then \(W \subseteq V\) implies \(P[W] \subseteq P[V]\).

Proposition 3.3.3: Let \((A, \Lambda, V)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, .)\). Let \(W\) and \(V\) be any two nonempty finite subsets of \(A\). Then \(P[W \cup V] = P[W] \cup P[V]\).

3.4. Examples

3.4.1. Example: Let \(R' = R^+ \cup \{0\}\), where \(R^+\) is the set of all positive real numbers and let \(W = \{0,1,2,3,\ldots\}\) \((R', \leq)\) is a lattice in which \(\Lambda\) and \(V\) are defined by \(\Lambda a b = \min\{a, b\}\) and \(V a b = \max\{a, b\}\), for all \(a, b \in R'\).

Here \(ma\) is the usual multiplication of \(a\) by \(m\).

Clearly for each \(m \in W\), \(m \neq 0\), and for each \(a \in R'\), \(ma \in R'\).

Also,

(i) \(m(a \Lambda b) = ma \Lambda mb\)

(ii) \(m(a V b) = ma V mb\)

(iii) \(ma V na \leq (m + n)a\) and \(ma \Lambda na \leq (m + n)a\)

(iv) \((mn)a = m(na)\), for all \(m, n \in W\), \(m \neq 0\), \(n \neq 0\), and \(a, b \in R'\)

(v) \(1.a = a\), for all \(a \in R'\)

Therefore, \(R'\) is an Artex Space over the bi-monoid \((W, +, .)\).

Generally, if \(\Lambda_1, \Lambda_2,\) and \(\Lambda_3\) are the cap operations of \(A, B\) and \(C\) respectively and if \(V_1, V_2,\) and \(V_3\) are the cup operations of \(A, B\) and \(C\) respectively, then the cap of \(A \times B \times C\) denoted by \(\Lambda\) and the cup of \(A \times B \times C\) denoted by \(V\) are defined

\[ x \Lambda y = (a_1, b_1, c_1) \Lambda (a_2, b_2, c_2) = (a_1 \Lambda_1 a_2, a_3, b_1 \Lambda_2 b_2, a_3, c_1 \Lambda_3 c_2, a_3, c_2) \quad \text{and} \quad x V y = (a_1, b_1, c_1) V (a_2, b_2, c_2) = (a_1 V_1 a_2 V_1 a_3, b_1 V_2 b_2 V_2 b_3, c_1 V_3 c_2 V_3 c_3) \]

Here, \(\Lambda_1, \Lambda_2,\) and \(\Lambda_3\) denote the same meaning minimum of two elements in \(R'\) and \(V_1, V_2,\) and \(V_3\) denote the same meaning maximum of two elements in \(R'\).

Therefore, \(R'^3 = R' \times R' \times R'\) is an Artex Space over \(W\), where cap and cup operations are denoted by \(\Lambda\) and \(V\) respectively.

Let \(S = \{(1,0,0)\}\) and let \(T = \{(0,1,0)\}\)

Now \(P[S] = \{(m, 0, 0) / mcR'\}\) and \(P[T] = \{(0, n, 0) / ncR'\}\)

\[ P[S] V P[T] = \{(m, 0, 0) / mcR'\} V \{(0, n, 0) / ncR'\} = \{(m V_1 0, 0 V_2 n, 0 V_3 0)\} = \{(m, n, 0)\}\]

\[ P[S] V P[T] = \{(m, n, 0) / m, ncR'\} \quad \text{(i)} \]

Now \(S \cup T = \{(1, 0, 0), (0, 1,0)\}\)
Let \( m, n \in M, m \neq 0, n \neq 0 \)

Then \( m(1,0,0) \lor n(0,1,0) = (m,0,0) \lor (0,n,0) = (m \lor 1,0 \lor n,0) = (m,n,0) \) (since \( mV_10 = \max \{m,0\} = m, 0V_2n = \max \{0,n\} = n \) and \( 0V_30 = \max \{0,0\} = 0 \))

Therefore, \( P [S \cup T] = \{(m,n,0) / m, n \in R'\} \) \( \text{(ii)} \)

From equations (i) and (ii) we have \( P [S \cup T] = P [S] \lor P [T] \)

3.4.2. Example: Let \( S = \{(1, 0, 0)\} \) and let \( T = \{(1,0,0),(0,1,0)\} \)

Then \( P [S] = \{(a, 0, 0) / a \in R'\} \) and \( P [T] = \{(a, 0, 0), (0, b, 0) / a, b \in R'\} \)

Therefore, \( P [S] \subseteq P [T] \).

3.5. THE PRODUCT OF SUBSETS OF AN ARTEX SPACE OVER A BI-MONOID: Let \( (A, \Lambda, V) \) be an Artex Space over a bi-monoid \( (M, +, \cdot) \). Let \( S \) and \( T \) be subsets of the Artex Space \( A \). Then the product of \( S \) and \( T \) denoted by \( S \Lambda T \) is defined by \( S \Lambda T = \{s \Lambda t / s \in S \text{ and } t \in T\} \)

3.6. Product Combination: Let \( (A, \Lambda, V) \) be a Completely Bounded Artex Space over a bi-monoid \( (M, +, \cdot) \). Let \( a_1, a_2, a_3, \ldots a_n \in A \). Then any element of the form \( m_1a_1 \Lambda m_2a_2 \Lambda m_3a_3 \Lambda \ldots \Lambda m_na_n \), where \( m_i \in M \), is called a Product Combination or Meet Combination of \( a_1, a_2, a_3, \ldots a_n \).

3.7. The Product Span of a Subset of a Completely Bounded Artex Space over a Bi-monoid: Let \( (A, \Lambda, V) \) be a Completely Bounded Artex Space over a bi-monoid \( (M, +, \cdot) \) and \( W \) be a nonempty finite subset of \( A \). Then the Product Span of \( W \) or Meet Span of \( W \) denoted by \( P[W] \) is defined to be the set of all product combinations of elements of \( W \). That is, if \( W = \{a_1, a_2, a_3, \ldots a_n\} \), then \( P[W] = \{m_1a_1 \Lambda m_2a_2 \Lambda m_3a_3 \Lambda \ldots \Lambda m_na_n / m_i \in M\} \).

3.8. PROPOSITION

Proposition 3.8.1: Let \( (A, \Lambda, V) \) be a Completely Bounded Artex Space over a bi-monoid \( (M, +, \cdot) \). Let \( W \) and \( V \) be any two nonempty finite subsets of \( A \). Then \( P[W \cup V] = P[W] \Lambda P[V] \).

3.9. Example: Let \( R' = R^+ \cup \{0\} \), where \( R^+ \) is the set of all positive real numbers and let \( W = \{0,1,2,3,\ldots\} \) is a lattice in which \( \Lambda \) and \( V \) are defined by \( a \Lambda b = \min \{a, b\} \) and \( a V b = \max \{a, b\} \), for all \( a, b \in R' \).

Here \( ma \) is the usual multiplication of \( a \) by \( m \).

Clearly for each \( m \in W, m \neq 0, \) and for each \( a \in R', ma \in R' \).

Also,

\( (i) \) \( m(a \Lambda b) = ma \Lambda mb \)
\( (ii) \) \( m(a \lor b) = ma \lor mb \)
\( (iii) \) \( ma \Lambda na \leq (m+n)a \) and \( ma \lor na \leq (m+n)a \)
\( (iv) \) \( (mn)a = m(na) \), for all \( m, n \in W, m \neq 0, n \neq 0, \) and \( a, b \in R' \)
\( (v) \) \( 1.a = a \), for all \( a \in R' \)

Therefore, \( R' \) is an Artex Space over the bi-monoid \( (W, +, \cdot) \).

Generally, if \( \Lambda_1, \Lambda_2, \) and \( \Lambda_3 \) are the cap operations of \( A, B \) and \( C \) respectively and if \( V_1, V_2, \) and \( V_3 \) are the cup operations of \( A, B \) and \( C \) respectively, then the cap of \( A \times B \times C \) denoted by \( \Lambda \) and the cup of \( A \times B \times C \) denoted by \( V \) are defined

\[ x \Lambda y = (a_1,b_1,c_1) \Lambda (a_2,b_2,c_2) = (a_1 \Lambda_1 a_2, a_1 \Lambda_2 a_2, b_1 \Lambda_1 b_2, b_1 \Lambda_2 b_2, c_1 \Lambda_1 c_2, c_1 \Lambda_2 c_2) \]
\[ x V y = (a_1,b_1,c_1) V (a_2,b_2,c_2) = (a_1 V_1 a_2, a_1 V_2 a_2, b_1 V_1 b_2, b_1 V_2 b_2, c_1 V_1 c_2, c_1 V_2 c_2) \]

Here, \( \Lambda_1, \Lambda_2, \) and \( \Lambda_3 \) denote the same meaning minimum of two elements in \( R' \) and \( V_1, V_2, \) and \( V_3 \) denote the same meaning maximum of two elements in \( R' \)

Therefore, \( R'^3 = R' \times R' \times R' \) is an Artex Space over \( W \), where cap and cup operations are denoted by \( \Lambda \) and \( V \) respectively.

Let \( H = \{(1, 0, 0)\} \) and let \( T = \{(0, 1, 0)\} \)
Now \( P[H] = \{(m,0,0) / m \epsilon R' \} \) and \( P[T] = \{(0,n,0) / n \epsilon R' \} \)

\[
P[H] \Lambda P[T] = \{(m,0,0) / m \epsilon R' \} \cup \{(0,n,0) / n \epsilon R' \} = \{(0,0,0) / m_{\Lambda} = 0, 0 \Lambda n = 0 \text{ and } 0 \Lambda 0 = 0 \}
\]

\[
P[H] \Lambda P[T] = \{(0,0,0) \}
\]

(i)

Now \( H \cup T = \{(1,0,0), (0,1,0) \} \)

Let \( m, n \epsilon M, m \neq 0, n \neq 0 \)

Then \( m(1,0,0) \Lambda n(0,1,0) = (m,0,0) \Lambda (0,n,0) = (0,0,0) \text{ (since } m_{\Lambda} = 0, 0 \Lambda n = 0 \text{ and } 0 \Lambda 0 = 0) \)

Therefore, \( P[H \cup T] = \{(0,0,0)\} \)  

(ii)

From equations (i) and (ii) we have \( P[H \cup T] = P[H] \Lambda P[T] \)

IV. THE PRODUCT SPAN OF SUM SPAN OF A SUBSET OF A COMPLETELY BOUNDED ARTEX SPACE OVER A BI-MONOID

4.1. The Product Span of Sum Span a Subset of a Completely Bounded Artex Space over a Bi-monoid: Let \((A, \Lambda, V)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, .)\) and \(W\) be a nonempty subset of \(A\). Then the Sum Span of \(W\) or Join Span of \(W\) denoted by \(S[W]\) is defined to be \(S[W] = \{m_1a_1V m_2a_2V m_3a_3V ... V m_na_n / m_i \epsilon M \text{ and } a_i \epsilon W\}\). The Product Span of \(W\) or Meet Span of \(W\) denoted by \(P[W]\) is defined to be \(P[W] = \{m_1a_1 \Lambda m_2a_2 \Lambda m_3a_3 \Lambda ... \Lambda m_na_n / m_i \epsilon M \text{ and } a_i \epsilon W\}\). Then \(P[S[W]]\) is Product Span of the Sum span \(S[W]\).

4.1.1 Note: Every element \(x\) of \(P[S[W]]\) is of the following form:

\[
x = (m_{11}a_{11}V m_{12}a_{12}V ... V m_{1p}a_{1p}) \Lambda (m_{21}a_{21}V m_{22}a_{22}V ... V m_{2k}a_{2k}) \Lambda ... ... \Lambda (m_{r1}a_{r1}V m_{r2}a_{r2}V ... V m_{rq}a_{rq}),
\]

where \(a_{ij} \epsilon W\) and \(m_{ij} \epsilon M\).

4.2. PROPOSITIONS

Proposition: 4.2.1 Let \((A, \Lambda, V)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, .)\) and \(W\) be a nonempty subset of \(A\). Then \(S[W] \subseteq P[S[W]]\).

Proof: Let \((A, \Lambda, V)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, .)\) and \(W\) be a nonempty subset of \(A\).

Then \(S[W] = \{m_1a_1V m_2a_2V m_3a_3V ... V m_na_n / m_i \epsilon M \text{ and } a_i \epsilon W\}\).

Let \(x \epsilon S[W]\)

Then \(x = m_1a_1V m_2a_2V m_3a_3V ... V m_na_n\), where \(m_i \epsilon M\) and \(a_i \epsilon W\)

Now every element of \(P[S[W]]\) is of the form

\[
(m_{11}a_{11}V m_{12}a_{12}V ... V m_{1p}a_{1p}) \Lambda (m_{21}a_{21}V m_{22}a_{22}V ... V m_{2k}a_{2k}) \Lambda ... ... \Lambda (m_{r1}a_{r1}V m_{r2}a_{r2}V ... V m_{rq}a_{rq}),
\]

where \(a_{ij} \epsilon W\) and \(m_{ij} \epsilon M\).

Take \(m_{11} = m_1, m_{12} = m_2, ... , m_{1p} = m_n\) if \(p = n\)

Take \(m_{11} = m_1, m_{12} = m_2, ... , m_{1n} = m_{1n+1} = m_{1n+2} = m_{1n+3} = 0\) if \(p > n\)

Take \(m_{11} = m_1, m_{12} = m_2, ... , m_{1p} = m_{p+1} = m_{p+2} = m_{p+3} = ... = m_{1n} = m_n\) if \(p < n\)

and

Take \(a_{11} = a_1, a_{12} = a_2, ... , a_{1p} = a_n\) if \(p = n\)

Take \(a_{11} = a_1, a_{12} = a_2, ... , a_{1n} = a_n\) if \(p > n\)

Take \(a_{11} = a_1, a_{12} = a_2, ... , a_{1p} = a_p\) and \(a_{1p+1} = a_{p+1} = a_{p+2} = ... = a_{1n} = a_n\) if \(p < n\)

Also take \(m_{ij} = 0\), for \(i \geq 2\)

Then clearly \(x \epsilon P[S[W]]\)

Hence, \(S[W] \subseteq P[S[W]]\).
Proposition: 4.2.2 Let \((A, \Lambda, V)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, .)\) and \(W\) be a nonempty subset of \(A\). Then \(P[S[W]]\) is a SubArtex space of \(A\).

Proof: Let \((A, \Lambda, V)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, .)\)

Let \(W\) be a nonempty subset of \(A\).

The Sum Span of \(W\) denoted by \(S[W]\) is defined to be
\[
S[W] = \{m_1a_1Vm_2a_2Vm_3a_3V \ldots Vm_na_n / m_i \in M \text{ and } a_i \in W\}.
\]

The Product Span of \(W\) denoted by \(P[W]\) is defined to be
\[
P[W] = \{m_1a_1 \Lambda m_2a_2 \Lambda m_3a_3 \Lambda \ldots \Lambda m_na_n / m_i \in M \text{ and } a_i \in W\}.
\]

Then \(P[S[W]]\) is Product Span of the Sum span \(S[W]\).

Claim: \(P[S[W]]\) is a SubArtex space of \(A\).

Let \(x, y \in P[S[W]]\) and \(m, n \in M\).

Now every element of \(P[S[W]]\) is of the form
\[
(m_1a_1Vm_2a_2V \ldots Vm_pa_p) \Lambda (m_2a_2Vm_2a_2V \ldots Vm_pa_p) \Lambda \ldots \ldots \Lambda (m_na_nVm_na_nV \ldots Vm_na_n),
\]
where \(a_i \in W\) and \(m_i \in M\).

Since \((A, \Lambda, V)\) is a Completely Bounded Artex Space over the bi-monoid \((M, +, .)\), \(A\) contains the least and the greatest elements namely 0 and 1.

Therefore, \(m_j\) can necessarily be taken as 0.

Therefore, \(x\) and \(y\) are the combinations of products and sums of elements of \(W\)

Therefore, \(mx \Lambda ny\) is the combinations of products and sums of elements of \(W\) and \(mx V ny\) is the combinations of products and sums of elements of \(W\).

Therefore, \(mx \Lambda ny \in P[S[W]]\) and \(mx V ny \in P[S[W]]\)

Hence, \(P[S[W]]\) is a SubArtex Space of \(A\).

V. CONCLUSION

Sum Combination, Sum Span, Product Combination, Product Span, Product Span of Sum Span of a subset of a Completely Bounded Artex space over a bi-monoid will motivate the researchers.

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