



# SYMMETRIC LEFT BI-DERIVATIONS ON SEMIPRIME RINGS

C. JAYA SUBBA REDDY\*<sup>1</sup>, G. VENKATABHASKARA RAO<sup>2</sup>

<sup>1,2</sup>Department of Mathematics,  
S. V. University, Tirupati – 517502, Andhra Pradesh, India.

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## ABSTRACT

Let  $R$  be a 2-torsion and 3-torsion free semiprime ring. Let  $D(.,.): R \times R \rightarrow R$  and  $B(.,.): R \times R \rightarrow R$  be a symmetric left bi-derivation and symmetric bi-additive mapping. If  $D(d(x), x) = 0$  and  $d(d(x)) = f(x)$  holds for all  $x$  in  $R$ , where  $d$  be a trace of  $D$  and  $f$  be a trace of  $B$ . In this case  $D = 0$ .

**Key Words:** Semiprime ring, Symmetric mapping, Trace, Symmetric bi-derivation, Symmetric bi-additive mapping, Symmetric left bi-derivation.

## INTRODUCTION

The concept of a symmetric bi-derivation has been introduced by Gy. Maksa in [2], [3]. A classical result in the theory of centralizing mappings is a theorem first proved by E. Posner [5]. J. Vukman [6] has studied some results concerning symmetric bi-derivations on prime and semi prime rings. In this paper we proved some results in symmetric left bi-derivations on semiprime rings.

Throughout this paper  $R$  will be associative. We shall denote by  $Z(R)$  the center of a ring  $R$ . Recall that a ring  $R$  is semiprime if  $aRa = (0)$  implies that  $a = 0$ . We shall write  $[x, y]$  for  $xy - yx$  and use the identities  $[xy, z] = [x, z]y + x[y, z]$ ,  $[x, yz] = [x, y]z + y[x, z]$ . An additive map  $d: R \rightarrow R$  is called derivation if  $d(xy) = d(x)y + xd(y)$  holds for all  $x, y \in R$ . A mapping  $B(.,.): R \times R \rightarrow R$  is said to be symmetric if  $B(x, y) = B(y, x)$  holds for all  $x, y \in R$ . A mapping  $f: R \rightarrow R$  defined by  $f(x) = B(x, x)$ , where  $B(.,.): R \times R \rightarrow R$  is a symmetric mapping, is called a trace of  $B$ . It is obvious that, in case  $B(.,.): R \times R \rightarrow R$  is symmetric mapping which is also bi-additive (i. e. additive in both arguments) the trace of  $B$  satisfies the relation  $f(x + y) = f(x) + f(y) + 2B(x, y)$ , for all  $x, y \in R$ . We shall use the fact that the trace of a symmetric bi-additive mapping is an even function. A symmetric bi-additive mapping  $D(.,.): R \times R \rightarrow R$  is called a symmetric bi-derivation if  $D(xy, z) = D(x, z)y + xD(y, z)$  is fulfilled for all  $x, y, z \in R$ . Obviously, in this case also the relation  $D(x, yz) = D(x, y)z + yD(x, z)$  for all  $x, y, z \in R$ . A symmetric bi-additive mapping  $D(.,.): R \times R \rightarrow R$  is called a symmetric left bi-derivation if  $D(xy, z) = xD(y, z) + yD(x, z)$  for all  $x, y, z \in R$ . Obviously, in this case also the relation  $D(x, yz) = yD(x, z) + zD(x, y)$  for all  $x, y, z \in R$ . A mapping  $f: R \rightarrow R$  is said to be commuting on  $R$  if  $[f(x), x] = 0$  holds for all  $x \in R$ . A mapping  $f: R \rightarrow R$  is said to be centralizing on  $R$  if  $[f(x), x] \in Z(R)$  is fulfilled for all  $x \in R$ . A ring  $R$  is said to be  $n$ -torsion free if whenever  $na = 0$ , with  $a \in R$ , then  $a = 0$ , where  $n$  is nonzero integer.

## MAIN RESULTS

**Lemma 1:** [4, Lemma 1] Let  $d: R \rightarrow R$  be a derivation, where  $R$  is a semiprime ring. Suppose that either

- (i)  $ad(x) = 0$ , for all  $x \in R$  or
- (ii)  $d(x)a = 0$ , for all  $x \in R$  holds. In both the cases we have  $a = 0$  or  $d = 0$ .

**Lemma 2:** [1, Lemma 3.10] Let  $R$  be a semiprime ring of characteristic not two and let  $a, b \in R$  be a fixed elements. If  $axb + bxa = 0$  is fulfilled for all  $x \in R$ , then either  $a = 0$  or  $b = 0$ .

**Theorem 1:** Let  $R$  be a 2-torsion free semiprime ring. Suppose there exists a symmetric left bi-derivation  $D(.,.): R \times R \rightarrow R$  such that  $D(d(x), x) = 0$  holds for all  $x \in R$ , where  $d$  be a trace of  $D$ . In this case  $D = 0$ .

**Proof:** We have  $D(d(x), x) = 0$ , for all  $x \in R$ .

(1)

*\*Corresponding Author: Dr. C. Jaya Subba Reddy\**

We replace  $d(x)$  by  $d(x)y$  in (1), we get

$$\begin{aligned} D(d(x)y, x) &= 0 \\ d(x)D(y, x) + yD(d(x), x) &= 0 \end{aligned}$$

By using (1) in the above equation, we get

$$\begin{aligned} d(x)D(y, x) &= 0 \\ d(x)D(x, y) &= 0, \text{ for all } x, y \in R. \end{aligned} \tag{2}$$

We replace  $x$  by  $x^2$  in (2), we get

$$\begin{aligned} d(x^2)D(x^2, y) &= 0 \\ 4x^2d(x)2xD(x, y) &= 0 \\ 8x^2d(x)xD(x, y) &= 0 \end{aligned}$$

$$\text{If } x = 0 \text{ it is trivial, if } x \neq 0 \text{ then } d(x)xD(x, y) = 0, \text{ for all } x, y \in R. \tag{3}$$

By the linearization of (1), we get

$$\begin{aligned} D(d(x+y), x+y) &= 0 \\ D(d(x) + d(y) + 2D(x, y), x+y) &= 0 \\ D(d(x), x) + D(d(x), y) + D(d(y), x) + D(d(y), y) + D(2D(x, y), x) + D(2D(x, y), y) &= 0 \end{aligned}$$

By using (1) in the above equation, we get

$$\begin{aligned} D(d(x), y) + D(d(y), x) + D(2D(x, y), x) + D(2D(x, y), y) &= 0 \\ D(d(x), y) + D(d(y), x) + 2D(D(x, y), x) + 2D(D(x, y), y) &= 0, \text{ for all } x, y \in R. \end{aligned} \tag{4}$$

We replace  $x$  by  $-x$  in (4), we get

$$\begin{aligned} D(d(-x), y) + D(d(y), -x) + 2D(D(-x, y), -x) + 2D(D(-x, y), y) &= 0 \\ D(d(x), y) - D(d(y), x) + 2D(D(x, y), x) - 2D(D(x, y), y) &= 0, \text{ for all } x, y \in R. \end{aligned} \tag{5}$$

By adding (4) and (5), we get

$$\begin{aligned} 2D(d(x), y) + 4D(D(x, y), x) &= 0 \\ D(d(x), y) + 2D(D(x, y), x) &= 0, \text{ for all } x, y \in R. \end{aligned} \tag{6}$$

We replace  $y$  by  $xy$  in (6), we get

$$\begin{aligned} D(d(x), xy) + 2D(D(x, xy), x) &= 0 \\ xD(d(x), y) + yD(d(x), x) + 2D(xD(x, y) + yD(x, x), x) &= 0 \\ xD(d(x), y) + yD(d(x), x) + 2D(xD(x, y), x) + 2D(yD(x, x), x) &= 0 \\ xD(d(x), y) + yD(d(x), x) + 2xD(D(x, y), x) + 2D(x, y)D(x, x) + 2yD(D(x, x), x) + 2D(x, x)D(y, x) &= 0 \\ xD(d(x), y) + yD(d(x), x) + 2xD(D(x, y), x) + 2D(x, y)d(x) + 2yD(d(x), x) + 2d(x)D(y, x) &= 0 \end{aligned}$$

By using (1) and (6) in the above equation, we get

$$\begin{aligned} 2D(x, y)d(x) + 2d(x)D(y, x) &= 0 \\ D(x, y)d(x) + d(x)D(x, y) &= 0, \text{ for all } x, y \in R. \end{aligned} \tag{7}$$

By using (2) in (7), we get

$$D(x, y)d(x) = 0, \text{ for all } x, y \in R. \tag{8}$$

We replace  $y$  by  $x$  in (7), we get

$$\begin{aligned} D(x, x)d(x) + d(x)D(x, x) &= 0 \\ d(x)d(x) + d(x)d(x) &= 0 \\ 2d(x)d(x) &= 0 \\ d(x)d(x) &= 0, \text{ for all } x \in R. \end{aligned} \tag{9}$$

We replace  $y$  by  $yx$  in (7), we get

$$\begin{aligned} D(x, yx)d(x) + d(x)D(x, yx) &= 0 \\ yD(x, x)d(x) + xD(x, y)d(x) + d(x)yD(x, x) + d(x)xD(x, y) &= 0 \\ yd(x)d(x) + xD(x, y)d(x) + d(x)yd(x) + d(x)xD(x, y) &= 0 \end{aligned}$$

By using (3), (8), (9) in above equation, we get

$$d(x)y d(x) = 0, \text{ for all } x, y \in R.$$

Which implies that  $d(x) = 0$ , for all  $x \in R$ , by semiprimeness of  $R$ , which means that  $D(x, y) = 0$ , for all  $x, y \in R$ .

**Theorem 2:** Let  $R$  be a 2-torsion and 3-torsion free semiprime ring. Let  $D(.,.): R \times R \rightarrow R$  and  $B(.,.): R \times R \rightarrow R$  be a symmetric left bi-derivation and symmetric bi-additive mapping respectively. Suppose that  $d(d(x)) = f(x)$  holds for all  $x \in R$ , where  $d$  be a trace of  $D$  and  $f$  be a trace of  $B$ . In this case  $D = 0$ .

**Proof:** We have  $d(d(x)) = f(x)$ , for all  $x \in R$ . (10)

By the linearization of (10), we get

$$\begin{aligned} d(d(x+y)) &= f(x+y) \\ d(d(x) + d(y) + 2D(x,y)) &= f(x) + f(y) + 2B(x,y) \\ d(d(x)) + d(d(y)) + d(2D(x,y)) + 2D(d(x), d(y)) + 2D(d(x), 2D(x,y)) + 2D(d(y), 2D(x,y)) \\ &= f(x) + f(y) + 2B(x,y) \\ d(d(x)) + d(d(y)) + 4d(D(x,y)) + 2D(d(x), d(y)) + 4D(d(x), D(x,y)) + 4D(d(y), D(x,y)) \\ &= f(x) + f(y) + 2B(x,y) \end{aligned}$$

By using (10) in the above equation, we get

$$\begin{aligned} 4d(D(x,y)) + 2D(d(x), d(y)) + 4D(d(x), D(x,y)) + 4D(d(y), D(x,y)) &= 2B(x,y) \\ 2d(D(x,y)) + D(d(x), d(y)) + 2D(d(x), D(x,y)) + 2D(d(y), D(x,y)) &= B(x,y), \text{ for all } x, y \in R. \end{aligned} \quad (11)$$

We replace  $x$  by  $-x$  in (11), we get

$$\begin{aligned} 2d(D(-x,y)) + D(d(-x), d(y)) + 2D(d(-x), D(-x,y)) + 2D(d(y), D(-x,y)) &= B(-x,y) \\ 2d(D(x,y)) + D(d(x), d(y)) - 2D(d(x), D(x,y)) - 2D(d(y), D(x,y)) &= -B(x,y), \text{ for all } x, y \in R. \end{aligned} \quad (12)$$

Subtract (12) from (11), we get

$$\begin{aligned} 4D(d(x), D(x,y)) + 4D(d(y), D(x,y)) &= 2B(x,y) \\ 2D(d(x), D(x,y)) + 2D(d(y), D(x,y)) &= B(x,y), \text{ for all } x, y \in R. \end{aligned} \quad (13)$$

We replace  $x$  by  $2x$  in (13), we get

$$\begin{aligned} 2D(d(2x), D(2x,y)) + 2D(d(y), D(2x,y)) &= B(2x,y) \\ 16D(d(x), D(x,y)) + 4D(d(y), D(x,y)) &= 2B(x,y) \\ 8D(d(x), D(x,y)) + 2D(d(y), D(x,y)) &= B(x,y), \text{ for all } x, y \in R. \end{aligned} \quad (14)$$

Subtract (13) from (14), we get

$$6D(d(x), D(x,y)) = 0$$

Since  $R$  is 2-torison and 3-torison free ring, we get

$$D(d(x), D(x,y)) = 0, \text{ for all } x, y \in R. \quad (15)$$

By using (15) and (13), we get

$$B(x,y) = 0, \text{ for all } x, y \in R.$$

We replace  $y$  by  $x$  in the above equation, we get  $f(x) = 0$ , for all  $x \in R$ . (16)

By using (1) and (16), we get

$$d(d(x)) = 0, \text{ for all } x \in R. \quad (17)$$

We replace  $y$  by  $yz$  in (15), we get

$$\begin{aligned} D(d(x), D(x,yz)) &= 0 \\ D(d(x), yD(x,z) + zD(x,y)) &= 0 \\ D(d(x), yD(x,z)) + D(d(x), zD(x,y)) &= 0 \\ yD(d(x), D(x,z)) + D(x,z)D(d(x), y) + zD(d(x), D(x,y)) + D(x,y)D(d(x), z) &= 0 \end{aligned}$$

By using (15) in the above equation, we get

$$D(x,z)D(d(x), y) + D(x,y)D(d(x), z) = 0, \text{ for all } x, y, z \in R. \quad (18)$$

We replace  $z$  by  $d(x)$  in (18), we get

$$\begin{aligned} D(x, d(x))D(d(x), y) + D(x,y)D(d(x), d(x)) &= 0 \\ D(x, d(x))D(d(x), y) + D(x,y)d(d(x)) &= 0 \end{aligned}$$

By using (17) in the above equation, we get

$$D(x, d(x))D(d(x), y) = 0, \text{ for all } x, y \in R. \quad (19)$$

We replace  $y$  by  $xy$  in (19), we get

$$D(x, d(x))D(d(x), xy) = 0$$

$$D(d(x), x)(xD(d(x), y) + yD(d(x), x)) = 0$$

$$D(d(x), x)xD(d(x), y) + D(d(x), x)yD(d(x), x) = 0$$

We replace  $y$  by  $x$  in the above equation we get  $D(d(x), x)xD(d(x), x) = 0$ , which implies  $D(d(x), x) = 0$  for all  $x \in R$  since we have assumed that  $R$  is semiprime. Now Theorem 1 completes the proof.

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