REVERSE VERTEX- MAGIC LABELING
OF GENERALIZED PETERSEN GRAPHS AND CONVEX POLYTOPES

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(Received On: 21-07-15; Revised & Accepted On: 06-08-15)

ABSTRACT
A reverse vertex – magic labeling on a graph with \( V \) vertices and \( E \) edges is a one-to-one map taking the vertices and edges on to the integers 1, 2, 3,……. \( V+E \) with the property that subtract the label on the vertex and the labels of its incident edges is constant, independent of the choice of the vertex. We give the reverse vertex –magic labeling for Generalized Petersen Graphs and Convex Polytopes.

1. INTRODUCTION
All graphs considered in this paper are finite, simple and undirected. The graph \( G \) has vertex set \( V = V(G) \) and edge set \( E = E(G) \) and we let \( v = |V| \) and \( e = |E| \). A general reference for graph theoretic notations in [9]. In this paper, we will deal only with connected graphs, although the concepts apply equally to graphs with more than one connected component.

A labeling for a graph is a map that takes graph elements to numbers. In this paper the domain is the set of all edges, giving rise to labeling.

In [8] we introduced the notion of a reverse super vertex labeling. This is an assignment of the integers from 1 to \( V+E \) to the vertices and edges of \( G \) so that at each vertex, the vertex label and the labels on the edges incident at that vertex subtract to a fixed constant. More formally, a one-to-one map \( \lambda \) from \( E \cup V \) to the integers \( \{1, 2, 3 \ldots \ V+E\} \) is a reverse vertex – magic labeling if there is a constant \( k \) so that for every vertex \( x \),

\[
 f(x) - \sum f(xy) = k
\]

Where the sum is over all vertices \( y \) adjacent to \( x \).

Set \( M = V + E \) and let \( s_v \) be the sum of the vertex labels and \( s_e \), the sum of the edge labels. Clearly since the labels are the numbers \( 1, 2, \ldots, M \), we have as the sum of all labels

\[
 s_v + s_e = \sum_{i=1}^{M} i = \binom{M+1}{2}
\]

At each vertex \( x_i \) we have

\[
 f(x) - \sum f(xy) = k
\]

We sum this over all \( V \) vertices \( x_i \). This adds each vertex label twice so that \( s_v - 2s_e = vk \)

\[
 s_v - 2s_e = vk
\]

Combining these two equalities gives us \( \binom{M+1}{2} - 3s_e = vk \)

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The edge labels are all distinct (as are all the vertex labels). The edges could conceivably receive the e smallest labels or at the other extreme, the e largest labels or anything between consequently we have

\[ \sum_{i=1}^{e} i \leq S_e \leq \sum_{i=1}^{M} i \] (4)

A singular results hold for \( S_e \). Combining (3) & (4), we get

\[
\begin{align*}
\left( e+1 \right) / 2 \leq & \left( M+1 \right) / 2 - \left( v+1 \right) / 2 \\
\left( e+1 \right) / 2 \leq & \left( M+1 \right) / 2 - \left( v+1 \right) / 2 \\
-3 \left( M+1 \right) / 2 \leq & -3 \left( e+1 \right) / 2 \\
3 \left( v+1 \right) / 2 \leq & 3 \left( e+1 \right) / 2 \\
3 \left( v+1 \right) / 2 \leq & 3 \left( e+1 \right) / 2 \\
\end{align*}
\] (5)

Which will give the range of the feasible values for \( k \).

It is clear from the definition of reverse vertex-magic labeling that when \( k \) is given and the edge labels are known, then the vertex labels are determined. So the labeling is completely described by the edge labels. In this paper, we give new reverse vertex –magic labeling of several classes of graphs.

2. LABELINGS OF GENERALIZED PETERSEN GRAPHS

Suppose \( g \) is a reverse edge-labeling of a graph, that is, a one-to-one map from \( E \) onto the integers \{1, 2, ..., e\}. Then the weight \( w(x) \) of a vertex \( x \in V \) is defined as the sum of labels assigned to all edges incident to \( x \).

An edge labeling \( g \) is said to be consecutive if the weights of all vertices constitute a set of consecutive integers. Two edge – labeling \( g \) and \( g' \) are said to be complementary if there is a constant \( c \) such that \( w_g(x) + w_{g'}(x) = c \) for all \( x \in V \).

Let \( I = \{1, 2, ..., n\} \) be an index set. For simplicity we use the convention that \( x_{j+n} = x_j \), for \( j = 1, 2, ..., 6 \).

A generalized Petersen graph \( P(n, m) \), \( 1 \leq m \leq n \), consists of an outer \( n \)-cycle \( y_1, y_2, ..., y_n \), a set of \( n \) spokes \( y_ix_i \), \( 1 \leq i \leq n \), and \( n \) inner edges \( x_i x_{i+m} \), \( 1 \leq i \leq n \), with indices taken modulo \( n \). The standard Petersen graph is the instance \( P(5,2) \).

\( P(n,m) \) is regular of degree 3 and has \( v = 2n \) vertices and \( e = 3n \) edges; thus \( M = 5n \).

Using (5), we can readily determine the feasible values of \( k \) for the generalized Petersen graphs \( P(n, m) \):

\[ -\frac{19n}{2} \leq k \leq -\frac{n}{2} - 1 \]

It was shown in [1] that for \( n \geq 4 \), \( n \) even and \( 1 \leq m \leq n - 1 \) the generalized Petersen graph \( P(n, m) \) has a consecutive reverse edge-magic labeling defined by the bijective mapping \( g_i: E[P(n, m)] \rightarrow \{1, 2, ..., 3n\} \).

\[ g(y_{j}, y_{j+1}) = \frac{5n+i+1}{2} \delta(i) + \frac{n-i+2}{2} \]

\[ g_i(y_i, x_i) = \frac{3n-i+1}{2} \delta(i) + \left[ \left( n+i-2 \right) \rho(i, 4) + \left( n+i-2 \right) \rho(6, i) \right] \delta(i+1) \]

\[ g_i(x_i, x_{i+m}) = \frac{3n+i+1}{2} \delta(i) + \left[ \left( 2n-i+3 \right) \rho(i, 4) + \left( 5n-i+3 \right) \rho(6, i) \right] \delta(i+1) \]
for $i \in I$ where

$$
\delta(x) = \begin{cases} 
0 & \text{if } x \equiv 0 \pmod{2} \\
1 & \text{if } x \equiv 1 \pmod{2} 
\end{cases}
$$

$$
\rho(x, y) = \begin{cases} 
0 & \text{if } x > y \\
1 & \text{if } x \leq y 
\end{cases}
$$

The weight of vertices under the mapping $g_1$ constitute the sets

$$
W_1 = \left\{ w_{g_1}(y_i) : i \in I \right\} = \left\{ \frac{7n}{2} + 2, \frac{7n}{2} + 3, \ldots, \frac{9n}{2} + 1 \right\}
$$

and

$$
W_2 = \left\{ w_{g_2}(x_i) : i \in I \right\} = \left\{ \frac{9n}{2} + 2, \frac{9n}{2} + 3, \ldots, \frac{11n}{2} + 1 \right\}
$$

Label the edges of the generalized Petersen graph $P(n, m)$ by the consecutive labeling $g_1$. If $g_2$ is the complementary vertex labeling with values in the set

$$
\left\{ |E| + 1, |E| + 2, \ldots, |E| + |V| \right\} = \left\{ 3n + 1, 3n + 2, \ldots, 5n \right\}
$$

then the labelings $g_1$ and $g_2$ combine to give a reverse vertex – magic labeling of $P(n, m)$. Since the largest labels are assigned to the vertices, it is easily seen that the resulting magic constant $k = -\frac{n}{2} - 1$ is the largest possible.

Thus we have the following

**Theorem 1:** For $n \geq 4$, $n$ even and $1 \leq m \leq \frac{n}{2} - 1$, the generalized Petersen graph $P(n, m)$ has a reverse vertex – magic labeling with $k = -\frac{n}{2} - 1$.

We know that if a regular graph $G$ possesses a reverse vertex – magic total labeling $\lambda$ with magic constant $k$, then $G$ also has a dual labeling $\lambda'$ having magic constant $k' = (1-r)(M+1)-k$, where $r$ is the number of edges incident to the vertex. Since $P(n,m)$ is regular, it has a dual labeling. Hence

**Corollary 1:** For $n \geq 4$, $n$ even and $1 \leq m \leq \frac{n}{2} - 1$, the generalized Petersen graph $P(n, m)$ has a reverse vertex –magic labeling with $k = -\frac{19n}{2} - 1$.

**Proof:** In this case the dual labeling $g'$ is defined by

- $g'_1(u) = |E| + |V| + 1 - g_1(u)$ for any edge $u \in E(P(n, m))$,
- $g'_2(x) = |E| + |V| + 1 - g_2(x)$ for any vertex $x \in V(P(n, m))$.

Since the magic constant for $g$ is $k = -\frac{n}{2} - 1$, then $g'$ is a reverse vertex–magic labeling with dual magic $k' = -\frac{19n}{2} - 1$.

**Definition 2:** The Prism $D_n$, $n \geq 3$, is a trivalent graph which can be defined as the Cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on $n$ vertices. We note that the prism $D_n$ is the generalized Petersen graph $P(n,1)$.

**Corollary 2:** For $n \geq 4$, $n$ even, the Prism $D_n$ has a reverse vertex –magic labeling with $k = -\frac{n}{2} - 1$ and another one with $k = -\frac{19n}{2} - 1$.
3. LABELINGS OF SOME FAMILIES OF CONVEX POLYTOPES

In this section, we shall investigate the graphs of two families of convex polytopes. First we consider the graphs $R_n$ consisting of $2n$ 5-sided faces, $n$ 6-sided faces and a pair of $n$-sided faces, embedded in the plane and labeled as in fig 2. Using equation (5) where $v=6n$ and $e=9n$, we can determine the feasible values of the magic constant $k$ for the graph $R_n = -\frac{57n}{2} - 1 \leq k \leq -\frac{3n}{2} - 1$.

The dual graph of $R_n$ is a planar graph $B_n$ which has been investigated in [2]. There it was shown that $B_n$ has a face anti-magic labeling $g$ i.e an edge labeling in which the sum around each face is a constant. To obtain a reverse vertex – magic labeling of $R_n$, we make use of the labeling $h_l = E(R_n) \rightarrow \{1, 2, 3, ..., 9\}$ which is a modification of the labeling $g$ from [2]. It is defined as follows (with $\delta$ and $\rho$ as defined in [6] and [7]):

- $h_l(x_{ij}, x_{i+1j}) = [(8n+i)\delta(i) + (n - i)\delta(i + 1)]\rho(i, n - 1) + n\rho(n, i)$
- $h_l(x_{ij}, x_{i+1j}) = \left(\frac{5n}{2} + i\right)\delta(i) + \left(\frac{5n}{2} - i + 1\right)\delta(i + 1)$
- $h_l(x_{i,j}, x_{i,j+1}) = \left(\frac{15n}{2} - i + 1\right)\delta(i) + \left(\frac{9n}{2} - i + 1\right)\delta(i + 1)$
- $h_l(x_{i,j}, x_{i,j+1}) = \left(\frac{13n}{2} + i\right)\delta(i) + \left(\frac{9n}{2} - i + 1\right)\delta(i + 1)$
- $h_l(x_{i,j}, x_{i+1,j}) = (n + 1)\rho(2i, 1) + \left[\frac{3n - i + 3}{2}\delta(i) + \left(\frac{9n}{2} + i\right)\delta(i + 1)\right]\rho(2, i)$
- $h_l(x_{i,j}, x_{i+1,j}) = \frac{11n + i + 1}{2}\delta(i) + \left(\frac{11n}{2} - i + 1\right)\delta(i + 1)$
- $h_l(x_{i,j}, x_{i,j+1}) = \frac{13n - i + 1}{2}\delta(i) + \frac{5n + i}{2}\delta(i + 1)$
The edge labeling $h_1$ is consecutive: the weights of vertices in turn assume the values
\[ \frac{21n}{2}, \frac{21n}{2} + 3, \ldots, \frac{33n}{2} + 1. \]

If $h_2$ is the complimentary vertex labeling with values in the set
\[ \{9n + 1, 9n + 2, \ldots, 15n\} \]
then the labeling $h_1$ and $h_2$ combine to give a reverse vertex- magic labeling of $R_n$ with the magic constant $k = -\frac{3n}{2} - 1$.

**Theorem 2:** For $n \geq 4$, $n$ even, the plane graph $R_n$ has a reverse vertex- magic labeling with $k = -\frac{3n}{2} - 1$.

Since $R_n$ is regular, we have a dual labeling as before.

**Corollary 3:** For $n \geq 4$, $n$ even, the plane graph $R_n$ has a reverse vertex- magic labeling with $k = -\frac{57n}{2} - 1$.

The final graphs we investigate are the antiprisms $A_n$, $n \geq 3$, a family of planar graphs that are regular degree 4. These are Archimedean convex polytopes and, in particular, $A_4$ is the octahedron.

We will denote the convex set of $A_n$ by $V = \{x_i: i \in I\} \cup \{y_i: i \in I\}$ and the edge set by
\[ E = \{(x_i, x_{i+1}): i \in I\} \cup \{(y_i, y_{i+1}): i \in I\} \cup \{(x_i, y_i): i \in I\} \cup \{(y_i, y_{i+1}): i \in I\} \]
as in figure 3.
From (5) we get the range of feasible values for $k$:

$$-15n - \frac{3}{2} \leq k \leq -3n - \frac{3}{2}$$

**Theorem 3:** For $n \geq 4$, $n$ even, the plane graph $A_n$ has a reverse vertex-magic labeling with $k = -9n$.

**Proof:** We construct an edge labeling $f_1$ of $A_n$, $n = 2m$, $m \geq 2$, in the following way:

- $f_1(x,x_{i+1}) = 6n \rho(i,1) + [(5n + i - 1) \delta(i) + i \delta(i + 1)] \rho(2,i)$
- $f_1(y,y_{i+1}) = [(5n + i) \delta(i) + (2n + i + 1) \delta(i + 1)] \rho(i,n - 1) + (2n + 1) \rho(n,i)$
- $f_1(x,y_i) = (3n + 1) \alpha(1,1,i) + (5n - 2i + 3) \alpha(2,i,m + 1) + (3n + 3) \alpha(m + 2,i,m + 2) + (5n - 2i + 3) \alpha(m + 3,i,n - 1) + (4n - 1) \alpha(n,i,n)$
- $f_1(y_i x_{i+1}) = 2n - 2i + 1$ Here $\alpha(x,y,z) = \begin{cases} 1 & \text{if } x \leq y \leq z \\ 0 & \text{otherwise} \end{cases}$

The edge labeling $f_1$ is a one-to-one map from $E(A_n)$ on to the set $\{i : i \in I\} \cup \{n+2j-1 : j = 1,2,...,2n\} \cup \{5n+i : i \in I\}$. The weights of the vertices under the edge labeling $f_1$ constitute the set

$$w = \{w_{f_1}(x) : x \in V(A_n)\} = \{10n + 2j : j = 1,2,3,...,2n\}$$

If $f_2$ is the complementary vertex labeling with values in the set $\{n+2j : j = 1,2,...,2n\}$ then the labeling $f_1$ and $f_2$ combine to give reverse vertex-magic labeling of $A_n$ with the magic constant $k = -9n$.

**Theorem 4:** For $n \geq 4$, $n$ even, the antiprism $A_n$ has a reverse vertex–magic labeling with $k = -9n$.

### 4. OPEN PROBLEMS

We have shown that there exist reverse vertex–magic labeling for the generalized Petersen graph $P(n, m)$ for $n \geq 4$, $n$ even. We conjecture that

**Conjecture 1:** There is a reverse vertex-magic labeling for the Prism $D_n = P(n,1)$ for all $n \geq 3$

More strongly,

**Conjecture 2:** There is a reverse vertex-magic labeling for the generalized Petersen graph $P(n, m)$ for all $n \geq 3$ and $2 \leq m \leq \frac{n}{2}$.

We have not yet found a construction that will produce a reverse vertex-magic labeling for the plane graph $R_n$ for $n$ odd. However, we suggest the following...
Conjecture 3: There is a reverse vertex-magic labeling for the plane graph $R_n$ for all $n \geq 3$.

Open Problem 1: Find a reverse vertex-magic labeling for the antiprism $A_n$ for all odd $n \geq 3$.

The Ladder $L_n$, $n \geq 3$, can be viewed as the Cartesian product $P_2 \times P_n$ of a path on two vertices and a path on $n$ vertices, or as a Prism $D_n$ with two edges deleted.

Open Problem 2: Find a reverse vertex-magic labeling for the Ladder $L_n$.

REFERENCES


Source of Support: Nil, Conflict of interest: None Declared

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