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FIXED POINT THEOREMS IN FUZZY METRIC SPACES

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ABSTRACT

In this present paper on fixed point theorems in fuzzy metric space. We extended to Fuzzy 2 – Metric space.

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1. INTRODUCTION

Now a day's some contractive condition is a central area of research on Fixed point theorems in fuzzy metric spaces satisfying. Zadeh [10] in 1965 was introduced fuzzy sets. After this developed and a series of research were done by several Mathematicians. Kramosil and Michlek [5] Helpern [4] in 1981 introduced the concept of fuzzy metric space in 1975 and fixed point theorems for fuzzy metric space. Later in 1994, A.George and P.Veeramani [3] modified the notion of fuzzy metric space with the help of t-norm. Fuzzy metric space, here we adopt the notion that, the distance between objects is fuzzy, the objects themselves may be fuzzy or not.

in this present papers Gahler [1], [2] investigated the properties of 2-metric space, and investigated contraction mappings in 2-metric spaces. We know that 2-metric space is a real valued function of a point triples on a set X, which abstract properties were suggested by the area function in the Eucledian space, The idea of fuzzy 2-metric space was used by Sushil Sharma [8] and obtained some fruitful results. prove some common fixed point theorem in fuzzy 2-metric space by employing the notion of reciprocal continuity, of which we can widen the scope of many interesting fixed point theorems in fuzzy metric space.

2. PRELIMINARY NOTES

Definition 2.1: A tiangular norm $*$ (shortly t– norm) is a binary operation on the unit interval [0, 1] such that for all a, b, c, d ∈ [0, 1] the following conditions are satisfied:

1. $a * 1 = a$;
2. $a * b = b * a$;
3. $a * b \le c * d$ whenever $a \le c$ and $b \le d$
4. $a * (b * c) = (a * b) * c$.

Example 2.2: Let (X, d) be a metric space. Define $a * b = ab$ (or $a * b = \min\{a, b\}$) and for all $x, y \in X$ and $t > 0$, $M(x,y,t) = \frac{t}{t + d(x,y)}$. Then (X, M, $*$) is a fuzzy metric space and this metric d is the standard fuzzy metric.

Definition 2.3: A sequence $\{x_n\}$ in a fuzzy metric space (X, M, $*$) is said

(i) To converge to x in X if and only if $M(x_n, x, t) = 1$ for each $t > 0$.
(ii) Cauchy sequence if and only if $M(x_{n+p}, x_n, t) = 1$ for each $p > 0$, $t > 0$.
(iii) to be complete if and only if every Cauchy sequence in X is convergent in X.

Definition 2.4: A pair (f, g) or (A, S) of self maps of a fuzzy metric space (X, M, $*$) is said

(i) To be reciprocal continuous if $\lim_{n\to\infty} fgx_n = fx$ and $\lim_{n\to\infty} gfx_n = gx$ whenever there exist a sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$.

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(ii) Semi-compatible if $\lim_{n\to\infty} ASx_n = Sx$ whenever there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ for some $x \in X$.

Definition 2.5: Two self maps A and B of a fuzzy metric space $(X, M, *)$ are said to be weak compatible if they commute at their coincidence points, that is $Ax = Bx$ implies $ABx = BAx$.

Definition 2.6: A pair (A, S) of self maps of a fuzzy metric space $(X, M, *)$ is said to be

Definition 2.7: A binary operation $*: [0, 1] \times [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if $([0, 1]), *)$ is an abelian topological monoid with unit 1 such that $a_1*b_1*c_1 \leq a_2*b_2*c_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$ for all a_1, a_2, b_1, b_2 and c_1, c_2 are in [0, 1].

Definition 2.8: A sequence $\{x_n\}$ in a fuzzy 2-metric space $(X, M, *)$ is said
(i) To converge to x in X if and only if $\lim_{n\to\infty} M(x_n, x, a, t) = 1$ for all $a \in X$ and $t > 0$.
(ii) Cauchy sequence, if and only if $\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1$ for all $a \in X$ and $p > 0$, $t > 0$.
(iii) To be complete if and only if every Cauchy sequence in X is convergent in X.

3. MAIN RESULTS

Common fixed point theorem in complete fuzzy metric space by employing the notion of reciprocal continuity. This result can be extended here to fuzzy 2-metric by Urmila Mishra *et.al* [9]

Theorem 3.1: Let A, B, S, T be self maps on a complete fuzzy 2-metric space $(X, M, *)$ where $*$ is a continuous t-norm, satisfying
(i) (T-1) $AX \subseteq TX$, $BX \subseteq SX$.
(ii) (T-2) (B, T) is weak compatible and reciprocal continuous,
(iii) (T-3) for each $x, y \in X$ and $t > 0$, $M(Ax, By, z, t) \geq M(Sx, Ty, z, t) * M(Ax, Sx, z, t) * M(By, Ty, z, t)$, where $\Phi : [0, 1] \to [0,1]$ is a continuous function such that $\Phi(1) = 1$, $\Phi(0) = 0$ and $\Phi(a) > a$ for each $0 < a < 1$.
If (A, S) is semicompatible and reciprocal continuous, then A, B, S, T have a unique common fixed point.

Proof: Since $AX \subset TX$ and $BX \subset SX$, for any $x0 \in X$, there exists $x_1 \in X$ such that $Ax_0 = Tx_1$ and for this $x_1 \in X$, there exists $x_2 \in X$ such that $Bx_1 = Sx_2$.

Inductively, we can find a sequence $\{y_n\}$ in X as follows:
$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for n = 1, 2……

From (iii),

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, z, t) &= M(Ax_{2n}, Bx_{2n+1}, z, t) \\ &\geq M(Sx_{2n}, Tx_{2n+1}, z, t) * M(Ax_{2n}, Sx_{2n}, z, t) * M(Bx_{2n+1}, Tx_{2n+1}, z, t) * M(Ax_{2n}, Tx_{2n+1}, z, t) \\ &= M(y_{2n}, y_{2n+1}, z, t) * M(y_{2n+1}, y_{2n}, z, t) * M(y_{2n+2}, y_{2n+1}, z, t) * M(y_{2n+1}, y_{2n+1}, z, t) \\ &\geq M(y_{2n}, y_{2n+1}, z, t) * M(y_{2n+1}, y_{2n+2}, z, t). \end{aligned}$$

we have that

$$M(y_{2n+1}, y_{2n+2}, z, t) \geq M(y_{2n}, y_{2n+1}, z, t). \tag{3.1.1}$$

Similarly, we have also

$$M(y_{2n+2}, y_{2n+3}, z, t) \geq M(y_{2n+1}, y_{2n+2}, z, t). \tag{3.1.2}$$

From (3.1.1) and (3.1.2), we have

$$M(y_{n+1}, y_{n+2}, z, t) \geq M(y_n, y_{n+1}, z, t). \tag{3.1.3}$$

From (3.1.3),

$$\begin{aligned} M(y_n, y_{n+1}, t) &\geq M(y_n, y_{n-1}, t/q) \geq M(y_{n-2}, y_{n-1}, t/q^2) \\ &\geq \ldots\ldots \geq M(y_1, y_2, t/q^n) \to 1 \text{ as } n\to\infty. \end{aligned}$$

So $M(y_n, y_{n+1}, z, t) \to 1$ as $n\to\infty$ for any $t > 0$.

For each $\varepsilon > 0$ and each $t > 0$, we can choose $n_0 \in N$ such that $M(y_n, y_{n+1}, t) > 1 - \varepsilon$ for all $n>n_0$.

For m, n ∈ N, we suppose $m \geq n$. Then we have that
$M(y_n, y_m, t) \geq M(y_n, y_{n+1}, t/m - n) * M(y_{n+1}, y_{n+2}, t/m - n) * \ldots * M(y_{m-1}, y_m, t/m - n)$

$$> \overbrace{(1-\varepsilon)*(1-\varepsilon)*\ldots*(1-\varepsilon)}^{m-n} \geq 1-\varepsilon$$ and hence $\{y_n\}$ is a Cauchy sequence in X.

Since (X, M, $*$) is complete, $\{y_n\}$ converges to some point z ∈ X, and so $\{Ax_{2n-2}\}$, $\{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converges to z.

$$ASx_{2n} \rightarrow Sz \quad (3.1.4)$$

and

$$Tx_{2n-1} \rightarrow Tz. \quad (3.1.5)$$

From (iv), we get
$M(ASx_{2n}, BTx_{2n-1}, z, t) \geq M(SSx_{2n}, TTx_{2n-1}, t) * M(ASx_{2n}, SSx_{2n}, z, t) * M(BTx_{2n-1}, TTx_{2n-1}, z, t) * M(ASx_{2n}, TTx_{2n-1}, z, t)$.

Taking limit as n→∞, and using (3.1.4) and (3.1.5),
$M(Sz, Tz, z, t) \geq M(Sz, Tz, z, t) * M(Sz, Sz, z, t) * M(Tz, Tz, z, t) * M(Sz, Tz, z, t)$
$\geq M(Sz, Tz, z, t) * 1 * M(Sz, Tz, z, t)$
$\geq M(Sz, Tz, z, t)$.

Thus we have
$M(Sz, Tz, z, t) \geq M(Sz, Tz, z, t)$, and hence Sz = Tz. (3.1.6)

Now, from (iv),
$M(Az, BTx_{2n-1}, z, t) \geq M(Sz, TTx_{2n-1}, z, t) * M(Az, Sz, z, t) * M(BTx_{2n-1}, TTx_{2n-1}, z, t) * M(Az, TTx_{2n-1}, z, t)$

which implies that taking limit as n→∞, and using (3.1.5), (3.1.6),
$M(Az, Tz, z, t) \geq M(Sz, Sz, z, t) * M(Az, Tz, z, t) * M(Tz, Tz, z, t) * M(Az, Tz, z, t)$
$\geq M(Az, Tz, z, t)$,
and hence
Az = Tz. (3.1.7)

From (3.1.6) and (3.1.7),
$M(Az, Bz, z, t) \geq M(Sz, Tz, z, t) * M(Az, Sz, z, t) * M(Bz, Tz, z, t) * M(Az, Tz, z, t)$
$= M(Az, Az, z, t) * M(Az, Az, z, t) * M(Bz, Az, z, t) * M(Az, Az, z, t)$
$\geq M(Az, Bz, z, t)$,
and so
Az = Bz. (3.1.8)

From (3.1.6), (3.1.7) and (3.1.8),
Az = Bz = Tz = Sz. (3.1.9)

Now, we show that Bz = z.
$M(Ax2n, Bz, z, t) \geq M(Sx2n, Tz, z, t) * M(Ax_{2n}, Sx_{2n}, z, t) * M(Bz, Tz, z, t) * M(Ax_{2n}, Tz, z, t)$

which implies that taking limit as n→∞, and using (3.1.6) and (3.1.7),
$M(z, Bz, z, t) \geq M(z, Tz, z, t) * M(z, z, z, t) * M(Bz, Tz, z, t) * M(z, Tz, z, t)$
$\geq M(z, Bz, z, t) * 1 * M(Az, Az, z, t) * M(z, Bz, z, t)$
$\geq M(z, Bz, z, t)$, and hence Bz = z.

Thus from (3.1.9), z = Az = Bz = Tz = Sz and z is a common fixed point of A, B, S and T.

For uniqueness, let w be another common fixed point of A, B, S and T. Then
$M(z, w, z, t) = M(Az, Bw, z, t)$
$\geq M(Sz, Tw, z, t) * M(Az, Sz, z, t) * M(Bw, Tw, z, t) * M(Az, Tw, z, t)$
$\geq M(z, w, z, t)$.

From, z = w. This completes the proof of theorem.

Corollary 3.2: [13] Let (X,M, $*$) be a complete fuzzy metric space and let A,B, S and T be self mappings of X satisfying (i) − (iii) of theorem 3.1 and there exists q ∈ (0, 1) such that
$M(Ax, By, z, t) \geq M(Sx, Ty, z, t) * M(Ax, Sx, z, t) * M(By, Ty, z, t) * M(By, Sx, 2z, t) * M(Ax, Ty, z, t)$ for every x, y ∈ X and t > 0.

Then A, B, S and T have a unique fixed point in X.

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