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ZERO-FREE REGION FOR POLYNOMIALS WITH RESTRICTED REAL COEFFICIENTS

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ABSTRACT

In this paper we prove some extension of the Eneström-Kakeya theorem says that. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree nsuch that* $0 < a_0 \le a_1 \le a_2 \le, \dots, \le a_n$ *then all the zeros of P(z) lie in |z|≤1. By relaxing the hypothesis of this result in several ways and obtain zero-free regions for polynomials with restricted coefficients and there by present some interesting generalizations and extensions of the Enestrom-Kakeya Theorem.*

Mathematics Subject Classification: *30C10, 30C15.*

Keywords: *Zeros of polynomial, Eneström-Kakeya theorem.*

1. INTRODUCTION

The well known Results Eneström-Kakeya theorem [1, 2] in theory of the distribution of zeros of polynomials is the following.

Theorem: (A_1) Let P(z)=$\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n* such that $0 < a_0 \le a_1 \le a_2 \le, \dots, \le a_n$ then all the zeros of P(z) lie in |z|≤1.

Applying the above result to the polynomial $z^n \mathrm{P}(\frac{1}{z})$ we get the following result:

Theorem: (A_2) *If P(z) =*$\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n such that* $0 < a_n \le a_{n-1} \le a_{n-2} \le, \dots, \le a_0$ *then P(z) does not vanish in |z|<1*

In the literature [3-8], there exist several extensions and generalizations of the Eneström-Kakeya Theorem.

In this paper we give generalizations of the above mentioned results. In fact, we prove the following results:

Theorem: 1 Let P(z) =$\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \ge m \ge 2$ with real coefficients such that $\rho \ge 0, k \ge 1$ *and* $ka_0 \ge a_1 \le a_2 \ge a_3 \le a_4 \ge \cdots \ge a_{n-m-1} \le a_{n-m} \ge a_{n-m+1} \ge \cdots \ge a_{n-2} \ge a_{n-1} \ge a_n - \rho$ if both n and (n-m) are even or odd

(Or)

$ka_0 \ge a_1 \le a_2 \ge a_3 \le a_4 \ge \cdots \le a_{n-m-1} \ge a_{n-m} \ge a_{n-m+1} \ge \cdots \ge a_{n-2} \ge a_{n-1} \ge a_n - \rho$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then

(i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{k(|a_0|+a_0)+|a_n|-(|a_0|+a_n)+2\rho+S_1}$ if both n and (n-m) are even or odd

where $S_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

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(ii) all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{k(|a_0|+a_0)+|a_n|-(|a_0|+a_n)+2\rho+S_2}$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $S_2 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2})\,]$

Corollary: 1 Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with positive real coefficients such that $\rho \geq 0, k \geq 1$ *and*

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \geq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if both n and (n-m) are even or odd

(Or)

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then

(i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{(2k-1)a_0+2\rho+S_1}$ if both n and (n-m) are even or odd

where $S_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

(ii) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{(2k-1)a_0+2\rho+S_2}$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $S_2 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2})\,]$

Remark: 1 By taking $a_i > 0\ for\ i = 0,1,2, \ldots, n,$in theorem 1, then it reduces to Corollary 1.

Theorem: 2 Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that $\rho \geq 0, 0 < r \leq 1$ *and*

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if both n and (n-m) are even or odd

(Or)

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \geq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then

(i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{|a_n|+|a_0|-a_n-r(a_0+|a_0|)+2\rho+T_1}$ if both n and (n-m) are even or odd

where $T_1 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2})\,]$

(ii) all the zeros of P(z) does not vanish in the disk $||z| < \frac{|a_0|}{|a_n|+|a_0|-a_n-r(a_0+|a_0|)+2\rho+T_2}$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $T_2 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-2} + a_{n-m}) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$.

Corollary: 2 Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with positive real coefficients such that $\rho \geq 0, 0 < r \leq 1$ *and*

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if both n and (n-m) are even or odd

(Or)

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \geq a_{n-m-1} \leq a_{n-m} \geq a_{n-m+1} \geq \cdots \geq a_{n-2} \geq a_{n-1} \geq a_n - \rho$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then

(i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{(1-2r)a_0+2\rho+T_1}$ if both n and (n-m) are even or odd

where $T_1 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2})\,]$

(ii) all the zeros of P(z) does not vanish in the disk $||z| < \frac{a_0}{(1-2r)a_0+2\rho+T_2}$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $T_2 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-2} + a_{n-m}) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$.

Remark: 2 By taking $a_i > 0\ for\ i = 0,1,2, \ldots, n.$ in theorem 2, then it reduces to Corollary 4.

Theorem: 3 Let LetP(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that $\rho \geq 0, k \geq 1$ *and*

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if both n and (n-m) are even or odd

(Or)

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then

(i) all the zeros of P(z) does not vanish in the disk $|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + U_2}$ if both n and (n-m) are even or odd

where $U_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

(ii) all the zeros of P(z) does not vanish in the disk $|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + U_2}$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $U_2 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m})\,]$

Corollary: 3 Let LetP(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with positive real coefficients such that $\rho \geq 0, k \geq 1$ *and*

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if both n and (n-m) are even or odd

(Or)

$ka_0 \geq a_1 \leq a_2 \geq a_3 \leq a_4 \geq \cdots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then

(i) all the zeros of P(z) does not vanish in the disk $|z + k - 1| < \frac{a_0}{ka_0 + 2a_n + 2\rho + U_2}$ if both n and (n-m) are even or odd

where $U_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

(ii) all the zeros of P(z) does not vanish in the disk $|z + k - 1| < \frac{a_0}{ka_0 + 2a_n + 2\rho + U_2}$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $U_2 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m})\,]$.

Remark: 3 By taking $a_i > 0\ for\ i = 0,1,2, \ldots, n$ in theorem 3, then it reduces to Corollary 3.

Theorem: 4 Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with real coefficients such that $\rho \geq 0, 0 < r \leq 1$ *and* $ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if both n and (n-m) are even or odd

(Or)

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then

(i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{|a_0|}{|a_0| + |a_n| + a_n - \mathrm{r}(|a_0| + a_0) + V_1}$ if both n and (n-m) are even or odd

where $V_1 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m})]$

(ii) all the zeros of P(z) does not vanish in the disk $||z| < \frac{|a_0|}{|a_0| + |a_n| + a_n - \mathrm{r}(|a_0| + a_0) + V_2}$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $V_2 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$.

Corollary: 4 Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq m \geq 2$ with positive real coefficients such that $\rho \geq 0, 0 < r \leq 1$ *and*

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \leq a_{n-m-1} \geq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if both n and (n-m) are even or odd

(Or)

$ra_0 \leq a_1 \geq a_2 \leq a_3 \geq a_4 \leq \cdots \geq a_{n-m-1} \leq a_{n-m} \leq a_{n-m+1} \leq \cdots \leq a_{n-2} \leq a_{n-1} \leq a_n + \rho$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then

(i) all the zeros of P(z) does not vanish in the disk $|z| < \frac{a_0}{2a_n+(1-2\mathrm{r})a_0+V_1}$ if both n and (n-m) are even or odd where $V_1 = 2[\,(a_1 + a_3 + \cdots + \ a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + \ a_{n-m-2} + a_{n-m})\,]$

(ii)) all the zeros of P(z) does not vanish in the disk $||z| < \frac{a_0}{2a_n+(1-2\mathrm{r})a_0+V_2}$ if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $V_2 = 2[\,(a_1 + a_3 + \cdots + \ a_{n-m-4} + a_{n-m-2}) - (a_2 + a_4 + \cdots + \ a_{n-m-3} + a_{n-m-1})\,]$.

.

Remark: 4 By taking $a_i > 0\ for\ i = 0,1,2,\ldots,n$, in theorem 4, then it reduces to Corollary 4.

2. PROOFS OF THE THEOREMS

Proof of the Theorem 1: Let P(z)= $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n.

Let us consider the polynomial $J(z) = z^n \mathrm{P}\left(\frac{1}{z}\right)$ and $R(z) = \ (z-1)J(z)$ so that

$$R(z) = (z-1)(\,a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1}z^{n-m+1} + a_m z^{n-m} + a_{m+1}z^{n-m-1} + \cdots + a_{n-1}z + a_n)$$

$$= a_0 z^{n+1} - \{(a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_{m-1} - a_m)z^{n-m+1} + (a_m - a_{m+1})z^{n-m} + \cdots + (a_{n-1} - a_n)z + a_n\}$$

Also if $|z| > 1$ then $\frac{\mathbf{1}}{|\boldsymbol{z}|^{\boldsymbol{n-i}}} < for\ i = 0,1,2,\ldots,n-1.$

Now

$$|R(z)| \geq |a_0||z|^{n+1} - \{\ \ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_{n-1} - a_n||z| + |a_n|\,\}$$

$$\geq |a_0||z|^n|z|^n[\,|\mathrm{z}| - \tfrac{1}{|a_0|}\{\,|a_0 - a_1| + \tfrac{|a_1-a_2|}{|z|} + \tfrac{|a_2-a_3|}{|z|^2} + \tfrac{|a_3-a_4|}{|z|^3} + \cdots + \tfrac{|a_{m-1}-a_m|}{|z|^{m-1}} + \tfrac{|a_m-a_{m+1}|}{|z|^m} + \cdots + \tfrac{|a_{n-3}-a_{n-2}|}{|z|^{n-3}} + \tfrac{|a_{n-2}-a_{n-1}|}{|z|^{n-2}} + \tfrac{|a_{n-1}-a_n|}{|z|^{n-1}} + \tfrac{|a_n|}{|z|^n}\}]$$

$$\geq |a_0||z|^n[|\mathrm{z}| - \tfrac{1}{|a_0|}\{|\,k\,a_0 - a_1 - ka_0 + a_0| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + \cdots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \cdots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} + \rho - a_n - \rho| + |\,a_n|\,\}]$$

$$\geq |a_0||z|^n|z|^n[\,|\mathrm{z}| - \tfrac{1}{|a_0|}\{(ka_0 - a_1) + (\mathrm{k}-1)|a_0| + (\,a_2 - a_1) + (\,a_2 - a_3) + \cdots + (\,a_{n-m} - a_{n-m-1}) + (\,a_{n-m} - a_{n-m+1}) + \cdots + (\,a_{n-3} - a_{n-2}) + (\,a_{n-2} - a_{n-1}) + (\,a_{n-1} + \rho - a_n) + \rho + |a_n|\,\}]$$ if both n and (n-m) are even or odd

$$= |a_0||z|^n[\,|\mathrm{z}| - \tfrac{1}{|a_0|}\{\,k(|a_0| + a_0) + \ |a_n| - (|a_0| + a_n) + 2\rho + S_1\}]$$

where $S_1 = 2[\,(a_2 + a_4 + \cdots + \ a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \cdots + \ a_{n-m-3} + a_{n-m-1})\,]$

$$\Rightarrow R(z) > 0\ if\ |z| > \frac{1}{|a_0|}\{\,k(|a_0| + a_0) + \ |a_n| - (|a_0| + a_n) + 2\rho + S_1\}$$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}\{\,k(|a_0| + a_0) + \ |a_n| - (|a_0| + a_n) + 2\rho + S_1\}$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|a_0|}\{\,k(|a_0| + a_0) + \ |a_n| - (|a_0| + a_n) + 2\rho + S_1\}$$

Since P(z) = z^nJ($\frac{1}{z}$) it followed by replacing z by $\frac{1}{z}$, all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{k(|a_0| + a_0) + \ |a_n| - (|a_0| + a_n) + 2\rho + S_1},$$

if both n and (n-m) are even or odd.

Hence all the zeros P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{k(|a_0| + a_0) + |a_n| - (|a_0| + a_n) + 2\rho + S_1}$$

if both n and (n-m) are even or odd
where $S_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-2} + a_{n-m}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

Similarly we can also prove for if n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros P(z) does not vanish in the disk.

$$|z| < \frac{|a_0|}{k(|a_0| + a_0) + |a_n| - (|a_0| + a_n) + 2\rho + S_2}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $S_2 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m-2})\,]$

This completes the proof of the Theorem 1.

Proof of the Theorem 2:

Let P(z)= $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n

Let us consider the polynomial $J(z) = z^n \mathrm{P}\left(\frac{1}{z}\right)$ and $R(z) = (z-1)J(z)$ so that

$$R(z) = (z-1)(\,a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m-1} + \cdots + a_{n-1} z + a_n)$$

$$= a_0 z^{n+1} - \{(a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{m-1} - a_m) z^{n-m+1} + (a_m - a_{m+1}) z^{n-m} + \cdots + (a_{n-1} - a_n) z + a_n\}$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < for\ i = 0,1,2,\dots,n-1.$

Now $|R(z)| \geq |a_0||z|^{n+1} - \{\ |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_{n-1} - a_n||z| + |a_n|\,\}$

$$\geq |a_0||z|^n|z|^n[\,|z| - \frac{1}{|a_0|}\{\,|a_0 - a_1| + \frac{|a_1-a_2|}{|z|} + \frac{|a_2-a_3|}{|z|^2} + \frac{|a_3-a_4|}{|z|^3} + \cdots + \frac{|a_{m-1}-a_m|}{|z|^{m-1}} + \frac{|a_m-a_{m+1}|}{|z|^m} + \cdots + \frac{|a_{n-3}-a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2}-a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1}-a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\}]$$

$$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{|\,ra_0 - a_1 - ra_0 + a_0| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + \cdots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \cdots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} + \rho - a_n - \rho| + |\,a_n|\,\}]$$

$$\geq |a_0||z|^n|z|^n[\,|z| - \frac{1}{|a_0|}\{(\,a_1 - ra_0) + (1-\mathrm{r})|a_0| + (\,a_1 - a_2) + (\,a_3 - a_2) + \cdots + (\,a_{n-m-1} - a_{n-m}) + (\,a_{n-m} - a_{n-m+1}) + \cdots + (\,a_{n-3} - a_{n-2}) + (\,a_{n-2} - a_{n-1}) + (\,a_{n-1} + \rho - a_n) + \rho + |a_n|\,\}]$$

if both n and (n-m) are even or odd

$$= |a_0||z|^n[\,|z| - \frac{1}{|a_0|}\{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1\}\,]$$

Where $T_1 = 2[\,(a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2})\,]$

$$\Rightarrow R(z) > 0\ if\ |z| > \frac{1}{|a_0|}\{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1\}$$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}\{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1\}$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \le \frac{1}{|a_0|}\{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1\}$$

Since P(z) = z^nJ($\frac{1}{z}$) it followed by replacing z by $\frac{1}{z}$, all the zeros of P(z) lie in

$$|z| \ge \frac{|a_0|}{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1},$$

if both n and (n-m) are even or odd.

Hence all the zeros P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_1}$$

if both n and (n-m) are even or odd
where $T_1 = 2[\,(a_1 + a_3 + \cdots + \ a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + \ a_{n-m-4} + a_{n-m-2})\,]$

Similarly we can also prove for if n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros P(z) does not vanish in the disk.

$$|z| < \frac{|a_0|}{|a_n| + |a_0| - a_n - r(a_0 + |a_0|) + 2\rho + T_2}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even
where $T_2 = 2\big[(a_1 + a_3 + \cdots + \ a_{n-m-2} + a_{n-m})\text{–}(a_2 + a_4 + \cdots + \ a_{n-m-3} + a_{n-m-1})\big]$.

This completes the proof of the Theorem 2.

Proof of the Theorem 3: Let P(z)= $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n

Let us consider the polynomial $J(z) = z^n \mathrm{P}\left(\frac{1}{z}\right)$ and $R(z) = \ (z-1)J(z)$ so that

$$R(z) = (z-1)(\,a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1}z^{n-m+1} + a_m z^{n-m} + a_{m+1}z^{n-m-1} + \cdots + a_{n-1}z + a_n)$$

$$= a_0 z^n(z + k - 1) - \{(ka_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \cdots + (a_{m-1} - a_m)z^{n-m+1} + (a_m - a_{m+1})z^{n-m} + \cdots + (a_{n-1} - a_n)z + a_n\}$$

Also if $|z| > 1$ then$\frac{\mathbf{1}}{|\boldsymbol{z}|^{\boldsymbol{n-i}}} < for\ i = 0,1,2,\ldots,n-1.$

Now

$$|R(z)| \ge \ |a_0||z|^n|\mathrm{z} + \mathrm{k} - 1| - \{\ \ |ka_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_{n-1} - a_n||z| + |a_n|\,\}$$

$$\ge |a_0||z|^n|z|^n[\,|\mathrm{z} + \mathrm{k} - 1| - \tfrac{1}{|a_0|}\{\,|ka_0 - a_1| + \tfrac{|a_1 - a_2|}{|z|} + \tfrac{|a_2 - a_3|}{|z|^2} + \tfrac{|a_3 - a_4|}{|z|^3} + \cdots + \tfrac{|a_{m-1} - a_m|}{|z|^{m-1}} + \tfrac{|a_m - a_{m+1}|}{|z|^m} + \cdots + \tfrac{|a_{n-3} - a_{n-2}|}{|z|^{n-3}} + \tfrac{|a_{n-2} - a_{n-1}|}{|z|^{n-2}} + \tfrac{|a_{n-1} - a_n|}{|z|^{n-1}} + \tfrac{|a_n|}{|z|^n}\}]$$

$$\ge |a_0||z|^n[|\mathrm{z} + \mathrm{k} - 1| - \tfrac{1}{|a_0|}\{|ka_0 - a_1| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + \cdots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \cdots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} - \rho - a_n + \rho| + |\,a_n|\,\}]$$

$$\ge |a_0||z|^n|z|^n[\,|\mathrm{z} + \mathrm{k} - 1| - \tfrac{1}{|a_0|}\{(ka_0 - a_1) + (\,a_2 - a_1) + (\,a_2 - a_3) + \cdots + (\,a_{n-m} - a_{n-m-1}) + (\,a_{n-m+1} - a_{n-m}) + \cdots + (\,a_{n-2} - a_{n-3}) + (\,a_{n-1} - a_{n-2}) + (\,a_n + \rho - a_{n-1}) + \rho + |a_n|\,\}]$$

if both n and (n-m) are even or odd

$$= |a_0||z|^n[\,|\mathrm{z} + \mathrm{k} - 1| - \frac{1}{|a_0|}\{ka_0 + \ |a_n| + a_n + 2\rho + U_1\}]$$

where $U_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

$\Rightarrow R(z) > 0 \; if \; |z + k - 1| > \frac{1}{|a_0|}\{ka_0 + |a_n| + a_n + 2\rho + U_1\}$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z + k - 1| \le \frac{1}{|a_0|}\{ka_0 + |a_n| + a_n + 2\rho + U_1\}$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z + k - 1| \le \frac{1}{|a_0|}\{ka_0 + |a_n| + a_n + 2\rho + U_1\}$$

Since P(z) = z^n J($\frac{1}{z}$) it followed by replacing z by $\frac{1}{z}$,

all the zeros of P(z) lie in

$$|z + k - 1| \ge \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + U_1},$$

if both n and (n-m) are even or odd.

Hence all the zeros P(z) does not vanish in the disk

$$|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + U_1}$$

if both n and (n-m) are even or odd
where $U_1 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-4} + a_{n-m-2}) - (a_1 + a_3 + \cdots + a_{n-m-3} + a_{n-m-1})\,]$

Similarly we can also prove for if n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros P(z) does not vanish in the disk.

$$|z + k - 1| < \frac{|a_0|}{ka_0 + |a_n| + a_n + 2\rho + U_2}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $U_2 = 2[\,(a_2 + a_4 + \cdots + a_{n-m-3} + a_{n-m-1}) - (a_1 + a_3 + \cdots + a_{n-m-4} + a_{n-m})\,]$.

This completes the proof of the Theorem 3.

Proof of the Theorem 4: Let P(z)= $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree n

Let us consider the polynomial $J(z) = z^n$P($\frac{1}{z}$)

and $R(z) = (z-1)J(z)$ so that

$$R(z) = (z-1)(\,a_0 z^n + a_1 z^{n-1} + \cdots + a_{m-1} z^{n-m+1} + a_m z^{n-m} + a_{m+1} z^{n-m-1} + \cdots + a_{n-1} z + a_n)$$

$$= a_0 z^{n+1} - \{(a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{m-1} - a_m) z^{n-m+1} + (a_m - a_{m+1}) z^{n-m} + \cdots + (a_{n-1} - a_n) z + a_n\}$$

Also if $|z| > 1$ then $\frac{\mathbf{1}}{|z|^{n-i}} < for \; i = 0,1,2,\ldots,n-1$.

Now
$$|R(z)| \ge |a_0||z|^{n+1} - \{\, |a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + \cdots + |a_{m-1} - a_m||z|^{n-m+1} + |a_m - a_{m+1}||z|^{n-m} + \cdots + |a_{n-1} - a_n||z| + |a_n|\,\}$$

$$\geq |a_0||z|^n|z|^n[\,|z| - \frac{1}{|a_0|}\{\,|a_0 - a_1| + \frac{|a_1-a_2|}{|z|} + \frac{|a_2-a_3|}{|z|^2} + \frac{|a_3-a_4|}{|z|^3} + \cdots + \frac{|a_{m-1}-a_m|}{|z|^{m-1}} + \frac{|a_m-a_{m+1}|}{|z|^m}$$
$$+ \cdots + \frac{|a_{n-3}-a_{n-2}|}{|z|^{n-3}} + \frac{|a_{n-2}-a_{n-1}|}{|z|^{n-2}} + \frac{|a_{n-1}-a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\}]$$

$$\geq |a_0||z|^n[|z| - \frac{1}{|a_0|}\{|ra_0 - a_1 - ra_0 - a_0| + |a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4|$$
$$+ \cdots + |a_{m-1} - a_m| + |a_m - a_{m+1}| + \cdots + |a_{n-3} - a_{n-2}| + |a_{n-2} - a_{n-1}| + |a_{n-1} - a_n| + |\,a_n|\,\}]$$

$$\geq |a_0||z|^n|z|^n[\,|z| - \frac{1}{|a_0|}\{(\,a_1 - ra_0) + (1-\mathrm{r})|a_0| + (\,a_1 - a_2) + (\,a_3 - a_2)$$
$$+ \cdots + (\,a_{n-m-1} - a_{n-m}) + (\,a_{n-m+1} - a_{n-m}) + \cdots + (\,a_{n-2} - a_{n-3}) + (\,a_{n-1} - a_{n-2})$$
$$+(\,a_n - a_{n-1}) + |a_n|\,\}]$$

if both n and (n-m) are even or odd

$$= |a_0||z|^n[\,|z| - \frac{1}{|a_0|}\{|a_0| + |a_n| + a_n - \mathrm{r}(|a_0| + a_0) + V_1\}\,]$$

Where $V_1 = 2[\,(a_1 + a_3 + \cdots + \; a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + \; a_{n-m-2} + a_{n-m})\,]$

$$\Rightarrow R(z) > 0 \; if \; |z| > \frac{1}{|a_0|}\{|a_0| + |a_n| + a_n - \mathrm{r}(|a_0| + a_0) + V_1\}$$

This shows that all the zeros of R(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}\{|a_0| + |a_n| + a_n - \mathrm{r}(|a_0| + a_0) + V_1\}$$

But those zeros of R(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of R(z) and hence J(z) lie in

$$|z| \leq \frac{1}{|a_0|}\{|a_0| + |a_n| + a_n - \mathrm{r}(|a_0| + a_0) + V_1\}$$

Since P(z) = z^nJ($\frac{1}{z}$) it followed by replacing z by $\frac{1}{z}$,

all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{|a_0| + |a_n| + a_n - \mathrm{r}(|a_0| + a_0) + V_1},$$

if both n and (n-m) are even or odd.

Hence all the zeros P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_0| + |a_n| + a_n - \mathrm{r}(|a_0| + a_0) + V_1}$$

if both n and (n-m) are even or odd
where $V_1 = 2[\,(a_1 + a_3 + \cdots + \; a_{n-m-3} + a_{n-m-1}) - (a_2 + a_4 + \cdots + \; a_{n-m-2} + a_{n-m})\,]$

Similarly we can also prove for if n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute as above. That is all the zeros P(z) does not vanish in the disk.

$$|z| < \frac{|a_0|}{|a_0| + |a_n| + a_n - \mathrm{r}(|a_0| + a_0) + V_2}$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

where $V_2 = 2[\,(a_1 + a_3 + \cdots + \; a_{n-m-4} + a_{n-m-2}) - (a_2 + a_4 + \cdots + \; a_{n-m-3} + a_{n-m-1})\,]$.

This completes the proof of the Theorem 4.

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