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ON LINE-BLOCK GRAPHS

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ABSTRACT

In this paper, we introduce the concept of the line block graph of a graph. We establish some properties of this graph. Also characterizations are given for graphs G for which (i) line-block graph of G is a tree and (ii)the line-block graph of G and G are isomorphic. We establish some relationships between (i) line-block graph and line graph and (ii) line-block graph and block graph.

Keywords: *line-block graph, qlick graph, line graph, block graph.*

Mathematics Subject Classification: *05C10.*

1. INTRODUCTION

All graphs considered are finite, undirected without isolated points, loops or multiple lines. All definitions and notations not given in this paper may be found in Kulli [1].

If $B = \{u_1, u_2, ..., u_r, r \geq 2\}$ is a block of a graph G, then we say that point u_1 and block B are incident with each other, as are u_2 and B and so on. If $B = \{e_1, e_2, ..., e_s, s \geq 1\}$ is a block of a graph G, then we say that line e_1 and block B are incident with each other, as are e_2 and B and so on. If two distinct blocks B_1 and B_2 are incident with a common cutpoint, then they are adjacent blocks. This idea was introduced by Kulli in [2]. The points, lines and blocks of a graph are called its members.

The point-block graph $P_b(G)$ of a graph G is the graph whose point set is the set of points and blocks of G and two points are adjacent if the corresponding blocks are adjacent or the corresponding members are incident. This concept was introduced by Kulli and Biradar in [3] and was studied in [4, 5, 6]. Many other graph valued functions in graph theory were studied, for example, in [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

The qlick graph $Q(G)$ of a graph G is the graph whose point set is the set of lines and blocks of G and two points are adjacent if the corresponding lines and blocks are adjacent or the corresponding members are incident. This concept was introduced by Kulli in [23] and was studied in [24]. The block-line forest $B_l(G)$ of a graph G is the graph whose point set is the set of lines and blocks of G and two points are adjacent if the corresponding members are incident. This concept was introduced by Kulli in [25].

The block graph $B(G)$ of a graph G is the graph whose point set is the set of blocks of G and two points are adjacent if the corresponding blocks are adjacent. This concept was first studied by Harary in [26] and further this was studied by Kulli in [27, 28, 29]. The line graph $L(G)$ of a graph G is the graph whose point set corresponds to the lines of G such that two points of $L(G)$ are adjacent if the corresponding lines of G are adjacent. This graph was studied, for example, in [30, 31, 32, 33, 34, 35, 36].

The following will be useful in the proof of our results.

Theorem A [23]: If G is a nontrivial connected (p, q) graph whose points have degree d_i and if b_i is the number of blocks to which point v_i belongs in G, then the qlick graph $Q(G)$ has $q - p + \Sigma\, b_i + 1$ points and $\frac{1}{2}\sum d_i^2 + \frac{1}{2}\sum b_i\left(b_i - 1\right)$ lines.

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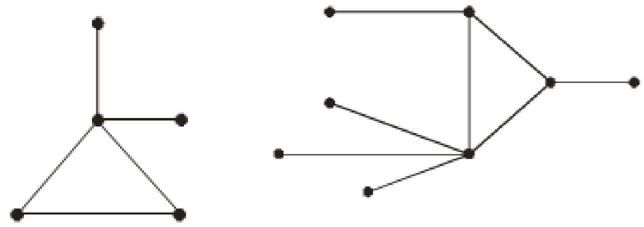
Theorem B [1, p.40]: If G is a (p, q) graph whose points have degree d_i, then the line graph $L(G)$ has q points and $\frac{1}{2}\sum d_i^2 - q$ lines.

2. LINE-BLOCK GRAPHS

The definition of the point-block graph $P_b(G)$ of a graph G inspired us to introduce the following graph valued function.

Definition 1: The line-block graph $L_b(G)$ of a graph G is the graph whose point set is the union of the set of lines and the set of blocks of G in which two points are adjacent if the corresponding blocks are adjacent or one corresponds to a block of G and other to a line incident with it.

Example 2: In Figure 1, a graph G and its line block graph $L_b(G)$ are shown.



G $\qquad$ $L_b(G)$

Figure-1

Remark 3: If G is a connected graph, then $L_b(G)$ is also a connected graph and conversely.

By definition, any point of G is not a point of $L_b(G)$. Thus we consider only graphs without isolated points.

Iterated line-block graphs are defined by $L_b^n(G) == L\left(L_b^{n-1}(G)\right)$ for $n \geq 2$ where $L_b^1(G) = L_b(G)$.

Remark 4: For any graph G, $L_b(G)$ is a spanning subgraph of $Q(G)$. Thus the graphs $L_b(G)$ and $Q(G)$ have the same number of points.

Remark 5: For any graph G, $B(G)$ is a subgraph of $L_b(G)$.

Remark 6: For any graph G, $B_l(G)$ is a subgraph of $L_b(G)$.

Remark 7: For any graph G, $Q(G) = L(G) \cup L_b(G)$.

The following theorem determines the number of points and lines in the line-block graph of a graph.

Theorem 8: If G is a nontrivial connected (p, q) graph whose points have degree d_i and if b_i is the number of blocks to which point v_i belongs in G, then the line block graph $L_b(G)$ of G has $q - p + \Sigma b_i + 1$ points and $q + \frac{1}{2}\sum b_i\left(b_i - 1\right)$ lines.

Proof: By Remark 4, the graphs $L_b(G)$ and $Q(G)$ have the same number of points. Hence by Theorem A, $L_b(G)$ has

$$q - p + \Sigma b_i + 1 \text{ points.}$$

by Remark 7, the number of lines of $Q(G)$ is the sum of the number of lines in $L(G)$ and in $L_b(G)$. By Theorem A the number of lines in $Q(G)$ is $\frac{1}{2}\sum d_i^2 + \frac{1}{2}\sum b_i\left(b_i - 1\right)$. Also by Theorem B, the number of lines in $L(G)$ is $-q + \frac{1}{2}\sum d_i^2$. Thus the number of lines in

$$\begin{aligned} L_b(G) &= \frac{1}{2}\sum d_i^2 + \frac{1}{2}\sum b_i\left(b_i - 1\right) - \left(-q + \frac{1}{2}\sum d_i^2\right) \\ &= q + \frac{1}{2}\sum b_i\left(b_i - 1\right). \end{aligned}$$

Corollary 9: Let G be a graph without isolated points. If G is a (p, q) graph with m components whose points have degree d_i and b_i is the number of blocks to which point v_i belongs in G, then the line-block graph $L_b(G)$ of G has $q-p+\sum b_i+m$ points and $q+\frac{1}{2}\sum b_i\left(b_i-1\right)$ lines.

Remark 10: For any block of G with at least 3 points, the corresponding point in $L_b(G)$ is a cut point of $L_b(G)$.

Remark 11: For any line of G, the corresponding point in $L_b(G)$ is an end point of $L_b(G)$.

Theorem 12: A graph G is a block if and only if the line-block graph $L_b(G)$ of G is a star.

Proof: Suppose G is a block. Then clearly $L_b(G)$ is a star.

Conversely suppose $L_b(G)$ is a star. We consider the following cases.

Case-1: Suppose $L_b(G) = K_{1, 1}$. Then G is $K_{1, 1}$.

Case-2: Suppose $L_b(G) = K_{1, p}$, $p \geq 2$. Then $L_b(G)$ has a unique cut point and by Remark 10, G has a unique block. It implies that G is itself a block.

From the above two cases, we see that G is a block.

Corollary 13: For any cycle C_p with $p \geq 3$ points, $L_b(C_p) = K_{1, p}$.

Corollary 14: For a complete graph K_p, $p \geq 2$, $L_b\left(K_p\right)=K_{1,\frac{p(p-1)}{2}}$.

Corollary 15: If G is a block with p lines, then $L_b(G) = K_{1, p}$.

Theorem 16: Let G be a nontrivial connected graph. The graphs G and $L_b(G)$ are isomorphic if and only if G is K_2.

Proof: Suppose G and $L_b(G)$ are isomorphic. We now prove that $G = K_2$. On the contrary, assume G is a connected graph with $p\geq3$ points. We now consider the following two cases.

Case-1: Suppose G is not a tree with $p\geq3$ points. Then G has at least p lines and has at least one block. Thus $L_b(G)$ has at least p+1 points. Therefore the number of points of G is less than that in $L_b(G)$. Hence G and $L_b(G)$ are not isomorphic, a contradiction.

Case-2: Suppose G is a tree with $p\geq3$ points. Then G has $p - 1$ lines and $p - 1$ blocks. Then $L_b(G)$ has $2p - 2$ points,

Thus the number of points of G is less than that in $L_b(G)$. Hence $G \neq L_b(G)$, a contradiction.

From the above two cases, we conclude that G is K_2.

Conversely suppose G is K_2. Obviously $G = L_b(G)$.

The following corollaries are immediate consequences of the above theorem.

Corollary 17: Let G be a nontrivial connected graph. Then $G=L_b^n\left(G\right)$, $n\geq1$, if and only if $G = K_2$.

Corollary 18: Let G be a graph without isolated points. Then $G=L_b^n\left(G\right)$, $n\geq1$, if and only if $G = mK_2$, $m\geq1$.

Theorem 19: Let G be a nontrivial connected graph. The line block graph $L_b(G)$ of G is a tree if and only if every point of G lies on at most 2 blocks.

Proof: Suppose $L_b(G)$ is a tree. We now show that every point of G lies on at most 2 blocks. Assume G has a point which lies on at least 3 blocks, say $b_1, ..., b_r$, $r\geq3$. It follows from definition, the corresponding points of $b_1, ..., b_r$, form K_r, $r\geq3$ as a subgraph of $L_b(G)$. Thus G contains a cycle, a contradiction. Hence every point of G lies on at most 2 blocks.

Conversely suppose every point of G lies on at most 2 blocks. We now consider the following two cases.

Case-1: Suppose every point of G lies on one block. Then G is a block. By Theorem 11, $L_b(G)$ is a star and hence $L_b(G)$ is a tree.

Case-2: Suppose a point of G lies on 2 blocks. It follows from definition, the corresponding points of blocks form K_2 as a subgraph and the corresponding point of a line which is in a block form an endline in $L_b(G)$. Therefore $L_b(G)$ has no cycles and hence $L_b(G)$ is a tree.

3. RELATION BETWEEN LINE-BLOCK GRAPH AND LINE GRAPH

Theorem 20: If G is a block with p lines, then $L(L_b(G)) = K_p$.

Proof: Suppose G is a block with p lines. By Corollary 14, $L_b(G) = K_{1,p}$. It is known that $L(K_{1,\,p}) = K_p$. Thus $L(L_b(G)) = K_p$.

4. RELATION BETWEEN LINE-BLOCK GRAPH AND BLOCK GRAPH

A graph G^+ is the end line graph of G if G^+ is obtained from G by adjoining an end line $u_i\,u_i'$ at each point u_i of G.

Proposition 21: If $G = K_{1,\,p},\, p \geq 2$, then $B(L_b(G))= G$.

Proof: Suppose G=$K_{1,\,p},\, p \geq 2$. Then $L_b\left(G\right) = K_p^+$.

We have $B\left(K_p^+\right) = K_{1,p}$. Thus $B\left(L_b\left(G\right)\right) = B\left(K_p^+\right) = K_{1,p}$. Therefore $B\left(L_b\left(G\right)\right) = G$.

Theorem 22: Let G be a nontrivial connected graph. Then $L_b(G)$ and $B(G)^+$ are isomorphic if and only if G is a tree.

Proof: Suppose G is a tree. Then every block of G is K_2. Thus there is a one-to-one correspondence between the points of $B(G)$ and blocks of G such that two points of $B(G)$ are adjacent if the corresponding blocks of G are adjacent. The graph $B(G)^+$ is obtained from $B(G)$ by adding a new line at each point of $B(G)$ such that this line has exactly one point in common with $B(G)$. By definition of $L_b(G)$, the points v_i, v_i' in $L_b(G)$ corresponding to line e_i and block b_i of G, respectively, are incident. By Remark 5, $B(G)$ is a subgraph of $L_b(G)$. In both $L_b(G)$ and $B(G)^+$, every point of the subgraph isomorphic to $B(G)$ is adjacent to exactly one end point. Hence $L_b(G)$ and $B(G)^+$ are isomorphic.

Conversely suppose $L_b(G) = B(G)^+$. We now prove that G is a tree. One the contrary, assume G has a cycle. Then the number of lines of G is greater than the number of blocks of G. Clearly $L_b(G)$ has less number of points than $B(G)^+$. Thus $L_b(G) \neq B(G)^+$, which is a contradiction. This completes the proof.

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