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# AN EQUIVALENCE RELATION: ROUGH SETS

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***ABSTRACT***

*In this present paper, we slightly deviate from the traditional setting to construct Rough sets through a novel equivalence relation by using Fuzzy sets. Also we construct a topology which we call a rough topology on the universe set and present a few examples in this context.*

***AMS Subject Classification:*** *03E72, 03E75.*

***Key Words:*** *Fuzzy set, $L-$Fuzzy set, Rough set, Upper Approximation, Lower Approximation, Information system, Equivalence relation.*

## INTRODUCTION

The problem of imperfect knowledge has been tackled for a long time by philosophers, logicians and mathematicians. Recently it became also a crucial issue for computer scientists, particularly in the area of artificial Intelligence. There are many approaches to the problem of how to understand and manipulate imperfect knowledge. The most successful approaches to tackle this problem are the Fuzzy set theory and the Rough set theory. Theories of fuzzy sets and rough sets are powerful mathematical tools for modeling various types of uncertainties.

In order to study the control problems of complicate systems and dealing with fuzzy information, American cyberneticist *L. A. Zadeh* introduced fuzzy set theory in his classical paper [6] of 1965. The idea and the concept of fuzzy set were introduced by *Zadeh* used the unit interval [0,1] to describe and deal with fuzzy phenomena. In 1967, J. *A. Goguen* [1] generalized this concept with $L-$ fuzzy sets.

A polish applied mathematician and computer scientist *Zdzislaw Pawlak* introduced rough set theory in his classical paper [2] of 1982. Rough set theory is a new mathematical approach to imperfect knowledge. This theory presents still another attempt to deal with the uncertainty and vagueness. The rough set theory has attracted the attention of many researchers and practitioners who contributed essentially to its development and application.

Rough sets have been proposed for a very wide variety of applications. In particular, the rough set approach seems to be important for artificial Intelligence and cognitive sciences, especially for machine learning, knowledge discovery, data mining, pattern recognition and approximate reasoning.

In this present work, we construct rough sets by defining an equivalence relation through fuzzy sets. Also we apply our theory to the real line.

In what follows $U$ and $\mathcal{P}(U)$ stand for the universe set and the collection of all subsets of $U$ (i.e., the power set) respectively. Let $\mathbb{R}$ denote the set of all real numbers.

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## 1. PRELIMINARIES

In this section, we present some basic definitions that are necessary for our present discussion.

**1.1 Definition:** Consider two sets $A$ and $B$, whose elements may be any objects whatsoever, and suppose that with each element $x$ of $A$ there is associated, in some manner, an element of $B$, which we denote by $f(x)$. Then $f$ is said to be a *function* or a *mapping* from $A$ to $B$.

**1.2 Definition:** A function $f: A \to B$ is said to be a *one-to-one* mapping from $A$ to $B$ if $f(x) \neq f(y)$ whenever $x \neq y$.

**1.3 Definition:** A function $f: A \to B$ is said to be *on-to* if for each $y \in B$, there exists $x \in A$ such that $f(x) = y$.

**1.4 Definition:** If there exists a one-to-one mapping of $A$ on-to $B$, we say that $A$ and $B$ can be put in *one-to-one correspondence*, or that $A$ and $B$ are *equivalent*, and we write $A \sim B$.

**1.5 Definition:** For any positive integer $n$, let $J_n = \{1, 2, 3, ..., n\}$. Let $\mathbb{N}$ denote the set of all positive integers. For any set $A$, we say that

**(a)** $A$ is *finite*, if $A \sim J_n$ for some $n$ (the empty set is also considered to be finite).

**(b)** $A$ is *infinite*, if $A$ is not finite.

**(c)** $A$ is *countable*, if $A \sim \mathbb{N}$.

**(d)** $A$ is *uncountable*, if $A$ is neither finite nor countable.

**1.6 Definition:** A relation $R$ on a non-empty set $S$ is said to be a *partially ordered relation* on $S$ if

(a) $xRx$ for all $x \in S$ (*reflexivity*)

(b) $xRy$ and $yRx \Rightarrow x = y$ (*anti-symmetry*)

(c) $xRy$ and $yRz \Rightarrow xRz$ (*transitivity*)

The pair $(S, R)$ is called a partially ordered set.

**1.7 Definition:** Let $(S, R)$ be a partially ordered set and $A \subset S$.

(a) An element $u \in S$ is called an *upper bound* of $A$ if $aRu \ \forall \ a \in A$.

(b) An element $u \in S$ is called a *lower bound* of $A$ if $uRa \ \forall \ a \in A$.

(c) An element $u \in S$ is called the least upper bound (*lub*) of $A$ if $u$ is an upper bound of $A$ and $uRv$ for any upper bound $v$ of $A$. We denote the *lub* of $A$ by the symbol $\vee A$.

(d) An element $u \in S$ is called the greatest lower bound (*glb*) of $A$ if $u$ is a lower bound of $A$ and $vRu$ for any lower bound $v$ of $A$. We denote the *glb* of $A$ by the symbol $\wedge A$.

**1.8 Definition:** A partially ordered set $(S, R)$ is said to be a *lattice* if each pair of elements $a, b \in S$ has both *lub* and *glb* in $S$. We denote the *lub* and *glb* of $a$ and $b$ by the symbols $a \vee b$ and $a \wedge b$ respectively.

**1.9 Definition:** A lattice $(S, R)$ is said to be a *complete lattice* if every infinite subset of $S$ has both *lub* and *glb* in $S$. We denote the *lub* and *glb* of a complete lattice $S$ by the symbols 0 and 1 respectively. They are called the zero element and all element of $S$.

**1.10 Remark:** Define a relation $\geq$ on $\mathcal{P}(U)$ as follows.

$A \geq B \Leftrightarrow B \subseteq A \quad for \quad A, B \in \mathcal{P}(U)$.

Clearly $(\mathcal{P}(U), \geq)$ is a complete lattice with zero element $0 = \phi$ and all element $1 = U$. For $A, B \in \mathcal{P}(U)$, $A \vee B = A \bigcup B$, $A \wedge B = A \bigcap B$.

**1.11 Definition:** An $L-$*fuzzy subset* on $U$ is a mapping $\mu: U \to L$, where $L$ is a complete lattice. The collection of all $L-$fuzzy subsets on $U$ is denoted by $L^U$ and it is called the $L-$fuzzy space.

**1.12Definition:** A relation $R$ on a non-empty set $S$ is said to be an *equivalence relation* on $S$ if

(a) $xRx$ for all $x \in S$ (*reflexivity*)

(b) $xRy \Leftrightarrow yRx$ (*symmetry*)

(c) $xRy$ and $yRz \Rightarrow xRz$ (*transitivity*)

We denote the equivalence class of an element $x \in S$ with respect to the equivalence relation $R$ by the symbol $R[x]$ and $R[x] = \{ y \in S : yRx \}$.

**1.13 Definition:** Let $X \subseteq U$. Let $R$ be an equivalence relation on $U$. Then we define the following.

(a) The *lower approximation* of $X$ with respect to $R$ is the set of all objects, which can be for certain classified as $X$ using $R$. That is the set $R_*(X) = \{ x : R[x] \subseteq X \}$.

(b) The *upper approximation* of $X$ with respect to $R$ is the set of all objects, which can be possibly classified as $X$ using $R$. That is the set $R^*(X) = \{ x : R[x] \cap X \neq \phi \}$.

(c) The *boundary region* of $X$ with respect to $R$ is the set of all objects, which can be classified neither as $X$ nor as not$-X$ using $R$. That is the set $\mathcal{B}_R(X) = R^*(X) - R_*(X)$.

**1.14 Definition:** A set $X \subseteq U$ is said to be a *rough set* with respect to an equivalence relation $R$ on $U$, if the boundary region

$$\mathcal{B}_R(X) = R^*(X) - R_*(X) \text{ is non-empty.}$$

## 2. CONSTRUCTION OF ROUGH SETS

In the Literature of Rough set theory, information systems are considered. An information system is a pair $(U, \mathcal{A})$ where $\mathcal{A}$ is a set of attributes. Each attribute $a \in \mathcal{A}$ is a mapping $a : U \to V_a$ where $V_a$ is the range set of the attribute $a \in \mathcal{A}$. Corresponding to each attribute $a \in \mathcal{A}$, an equivalence relation $R_a$ is defined on $U$ such that $xR_a y \Leftrightarrow a(x) = a(y)$. Rough sets are constructed through this relation as usual.

In this section, we slightly deviate from the above traditional setting to construct rough sets. We take $L = (\mathcal{P}(U), \geq)$, the complete lattice mentioned in the Remark – 1.10.

**1.15Definition:** Fix $\mu \in L^U$ and define a relation ${}_\mu R$ on $U$ as follows.

For $x, y \in U$, $x \,{}_\mu R\, y \Leftrightarrow \mu(x) \sim \mu(y)$. ( i.e., $\mu(x)$ and $\mu(y)$ are equivalent).

**1.16Proposition:** ${}_\mu R$ is an equivalence relation on $U$.

**1.17Remark:** If $x \in U$, then the equivalence class of $x$ under the equivalence relation ${}_\mu R$ is given by ${}_\mu R[x] = \{ y \in U : \mu(y) \sim \mu(x) \}$.

**1.18Definition:** The lower and upper approximations of a subset $X$ of $U$ under ${}_\mu R$ are as follows.

$${}_\mu R_*(X) = \{ x : {}_\mu R[x] \subseteq X \}$$

$${}_\mu R^*(X) = \{ x : {}_\mu R[x] \cap X \neq \phi \}$$

The boundary region of $X$ under ${}_\mu R$ is given by $\mathcal{B}_{{}_\mu R}(X) = {}_\mu R^*(X) - {}_\mu R_*(X)$.

**1.19Definition:** A subset $X$ of $U$ is a rough subset of $U$ if $\mathcal{B}_{{}_\mu R}(X)$ is non-empty.

**1.20Proposition:** Let $T_\mu = \{ X \in \mathcal{P}(U) : {}_\mu R_*(X) = X \}$. Then $T_\mu$ is a topology on $U$.

**1.21Remark:** We call the topology $T_\mu$, the rough topology with respect to the fuzzy set $\mu \in L^U$.

 

## 3. APPLICATIONS TO $\mathbb{R}$

**1.22Example:** Consider the null $L-$fuzzy set $\upsilon:\mathbb{R}\to\mathcal{P}(\mathbb{R})$ defined by $\upsilon(x)=\phi \quad \forall \quad x\in\mathbb{R}$ then ${}_{\upsilon}T=\{\phi,\mathbb{R}\}$, which is the indiscrete topology on $\mathbb{R}$.

**1.23 Example:** Consider the absolute $L-$fuzzy set $\lambda:\mathbb{R}\to\mathcal{P}(\mathbb{R})$ defined by $\lambda(x)=\mathbb{R} \quad \forall \quad x\in\mathbb{R}$ then ${}_{\lambda}T=\{\phi,\mathbb{R}\}$, which is the indiscrete topology on $\mathbb{R}$.

**1.24 Example:** Define $\mu:\mathbb{R}\to\mathcal{P}(\mathbb{R})$ as follows.

$$\mu(x)=\begin{cases}\mathbb{N} & if \quad x=0\\ \mathbb{R} & if \quad x\neq 0\end{cases}$$

Then $\mu$ is an $L-$Fuzzy set. Now consider the equivalence relation ${}_{\mu}\mathbb{R}$ on $\mathbb{R}$. Then

${}_{\mu}R[0]=\{x\in\mathbb{R}:\mu(x) \ \ is\ countable\}$

${}_{\mu}R[x]=\{y\in\mathbb{R}:\mu(y) \ \ is\ uncountable\}$ for any $x\neq 0$.

Clearly ${}_{\mu}R[0]=\mathbb{N} \ \ and \quad {}_{\mu}R[x]=\mathbb{R}-\{0\}$ for any $x\neq 0$.

In this case, we can observe the following.

(a) If $0\in X$ and $X\neq\mathbb{R}$ then ${}_{\mu}R_*(X)=\{0\}$.

(b) ${}_{\mu}R_*(\phi)=\phi={}_{\mu}R^*(\phi)$.

(c) ${}_{\mu}R_*(\mathbb{R})=\mathbb{R}={}_{\mu}R^*(\mathbb{R})$.

(d) If $0\notin X$ and $X\neq\mathbb{R}-\{0\}$ then ${}_{\mu}R_*(X)=\phi$.

(e) If $X=\mathbb{R}-\{0\}$ then ${}_{\mu}R_*(X)=\mathbb{R}-\{0\}$.

(f) If $X=\{0\}$ then ${}_{\mu}R^*(X)=\{0\}$.

(g) If $0\in X$ and $X\neq\{0\}$ then ${}_{\mu}R^*(X)=\mathbb{R}$.

(h) If $0\notin X$ then ${}_{\mu}R^*(X)=\mathbb{R}-\{0\}$.

Basing on the above facts, $T_{\mu}=\{\phi,\{0\},\mathbb{R}-\{0\},\mathbb{R}\}$ forms a topology on $\mathbb{R}$ and it is easy to observe that the topological space $(\mathbb{R},T_{\mu})$ is disconnected. If $\mathcal{R}_{\mu}$ is the collection of all rough sets in this space, then $\mathcal{R}_{\mu}=\mathcal{P}(\mathbb{R})-T_{\mu}$.

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