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APPROXIMATIONS OF SETS IN TOPOLOGICAL SPACES

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ABSTRACT

The classes of near open sets can be considered as rich sources for elementary concepts in approximation spaces. The purpose of this paper is to spot light on using some classes of near open sets as tools for measuring the exactness of sets.

Keywords: *Topological space, Topologized approximation space, Exact set, Internally definable set, Externally definable set, Rough set.*

1. PRELIMINARIES

A topological space [8] is a pair (X, τ) consisting of a set X and family τ of subsets of X satisfying the following conditions:
(T1) $\phi \in \tau$ and $X \in \tau$.
(T2) τ is closed under arbitrary union.
(T3) τ is closed under finite intersection.

Throughout this paper (X, τ) denotes a topological space, the elements of X are called points of the space, the subsets of X belonging to τ are called open sets in the space, the complement of the subsets of X belonging to τ are called closed sets in the space and the family of all τ - closed subsets of X is denoted by τ^*. The family τ of open subsets of X is also called a topology for X. A subset A of X in a topological space (X, τ) is said to be clopen if it is both open and closed in (X, τ).

A family $\mathcal{B} \subseteq \tau$ is called a base for a topological space (X, τ) iff every nonempty open subset of X can be represented as a union of subfamily of $\mathcal{B}$. Clearly, a topological space can have many bases. A family $S \subseteq \tau$ is called a subbase iff the family of all finite intersections of S is a base for (X, τ).

The τ - closure of a subset A of X is denoted by A^- and it is given by $A^- = \bigcap\{F \subseteq X : A \subseteq F$ and $F \in \tau^*\}$. Evidently, A^- is the smallest closed subset of X which contains A. Note that A is closed iff $A = A^-$. The τ - interior of a subset A of X is denoted by A^o and it is given by $A^o = \bigcup\{G \subseteq X: G \subseteq A$ and $G \in \tau\}$. Evidently, A^o is the largest open subset of X which contained in A. Note that A is open iff $A = A^o$.

Some forms of near open sets which are essential for our present study are introduced in the following definition.

Definition 1.1: Let (X, τ) be a topological space. The subset A of X is called:
(i) Semi-open [10] (briefly s-open) if $A \subseteq A^{o-}$.
(ii) Pre-open [12] (briefly p-open) if $A \subseteq A^{-o}$.
(iii) γ -open [7] (b-open [6]) if $A \subseteq A^{o-} \cup A^{-o}$.
(iv) α -open [13] if $A \subseteq A^{o-o}$.
(v) β -open [1] (Semi-pre-open [5]) if $A \subseteq A^{-o-}$.

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The complement of an *s*-open (resp. *p*-open, γ -open, α -open and β -open) set is called s -closed (resp. p -closed, γ - closed, α - closed and β -closed) set. The family of all *s*-open (resp. *p*-open, γ -open, α -open and β -open) sets of (X, τ) is denoted by $SO(X)$ (resp. $PO(X)$, $\gamma O(X)$, $\alpha O(X)$ and $\beta O(X)$). The family of all s -closed (resp. p - closed, γ - closed, α - closed and β - closed) sets of (X, τ) is denoted by $SC(X)$ (resp. $PC(X)$, $\gamma C(X)$, $\alpha C(X)$ and $\beta C(X)$).

The near interior (resp. near closure) of a subset A of X is denoted by $A^{j\,o}$ (resp. $A^{j\,-}$) and it is given by

$$A^{j\,o} = \bigcup \{G \subseteq X : G \subseteq A, G \text{ is a } j\text{-open set}\}$$

$$(\text{resp. } A^{j\,-} = \bigcap \{H \subseteq X : A \subseteq H, H \text{ is a } j\text{-closed set}\}), \text{ where } j \in \{ p, s, \gamma, \alpha, \beta \}.$$

From known results [1, 7], we have the following two remarks. The symbol " $\xrightarrow{\subseteq}$ " is used instead of " $\subseteq$ " in the implications between sets.

Remark 1.1: In a topological space (X, τ), the implications between τ and the families of near open sets are given in the following diagram.

$$\begin{tikzcd}
 & & SO(X) \arrow[dr, "\subseteq"] & & \\
\tau \arrow[r, "\subseteq"] & \alpha O(X) \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & \gamma O(X) \arrow[r, "\subseteq"] & \beta O(X) \\
 & & PO(X) \arrow[ur, "\subseteq"'] & &
\end{tikzcd}$$

Remark 1.2: In a topological space (X, τ), the implications between τ^* and the families of near closed sets are given in the following diagram.

$$\begin{tikzcd}
 & & SC(X) \arrow[dr, "\subseteq"] & & \\
\tau^* \arrow[r, "\subseteq"] & \alpha C(X) \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & \gamma C(X) \arrow[r, "\subseteq"] & \beta C(X) \\
 & & PC(X) \arrow[ur, "\subseteq"'] & &
\end{tikzcd}$$

Motivation for rough set theory has come from the need to represent subsets of a universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space $K = (X, R)$, where X is a set called the universe and R is an equivalence relation [11,14]. The equivalence classes of R are also known as the granules, elementary sets or blocks. We shall use R_x to denote the equivalence class containing $x \in X$ and X/R to denote the set of all elementary sets of R. In the approximation space $K = (X, R)$ the lower (resp. upper) approximation of a subset A of X is given by

$\underline{R}A = \{x \in X : R_x \subseteq A\}$ (resp. $\overline{R}A = \{x \in X : R_x \cap A \neq \phi\}$).

Pawlak noted [14] that the approximation space $K = (X, R)$ with equivalence relation R defines a uniquely topological space (X, τ) where τ is the family of all clopen sets in (X, τ) and X/R is a base of τ . Moreover, the lower (resp. upper) approximation of any subset A of X is exactly the interior (resp. closure) of A.

If R is a general binary relation, then the approximation space $K = (X, R)$ defines a uniquely topological space (X, τ_K), where τ_K is the topology associated to K (i.e τ_K is the family of all open sets in (X, τ_K) and $S = \{xR : x \in X\}$ is a subbase of τ_K , where $xR = \{ y \in X : x R y \}$) [4, 9].

Definition 1.2 [4]: Let $K=(X,R)$ be an approximation space with general binary relation *R* and τ_K is the topology associated to *K*. Then the triple $\kappa=(X,R,\tau_K)$ is called a topologized approximation space.

Definition 1.3 [4]: Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space and $A\subseteq X$. The lower (resp. upper) approximation of *A* is defined by

$$\underline{R}A = A^{o} \text{ (resp. } \overline{R}A = A^{-} \text{).}$$

Boundary region of a subset A of *X* [2] in a topologized approximation space $\kappa=(X,R,\tau_K)$ is denoted by $Bnd(A)$ and it is defined by $Bnd(A) = \overline{R}A - \underline{R}A$.

The following general definition is given to introduce the near lower and near upper approximations in a topologized approximation space $\kappa=(X,R,\tau_K)$.

Definition 1.4 [4]: Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space and $A\subseteq X$. The near lower (briefly *j*-lower) (resp. near upper (briefly *j*-upper)) approximation of *A* is denoted by $\underline{R}_jA$ (resp. $\overline{R}_jA$) and it is defined by

$\underline{R}_jA = A^{jo}$ (resp. $\overline{R}_jA = A^{j-}$), where $j\in\{p, s, \gamma, \alpha, \beta\}$.

Near boundary region of a subset A of *X* [2] in a topologized approximation space $\kappa=(X,R,\tau_K)$ is denoted by $Bnd_j(A)$ and it is defined by

$Bnd_j(A) = \overline{R}_jA - \underline{R}_jA$, where $j\in\{p, s, \gamma, \alpha, \beta\}$.

Remark 1.3 [2]: Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space and $A\subseteq X$. The implications between lower approximation and near lower approximations of *A* are given in the following diagram.

$$\begin{tikzcd}
 & & \underline{R}_sA \arrow[dr, "\subseteq"] & & \\
\underline{R}A \arrow[r, "\subseteq"] & \underline{R}_\alpha A \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & \underline{R}_\gamma A \arrow[r, "\subseteq"] & \underline{R}_\beta A \\
 & & \underline{R}_pA \arrow[ur, "\subseteq"'] & &
\end{tikzcd}$$

Remark 1.4 [2]: Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space and $A\subseteq X$. The implications between upper approximation and near upper approximations of *A* are given in the following diagram.

$$\begin{tikzcd}
 & & \overline{R}_sA \arrow[dr, "\subseteq"] & & \\
\overline{R}_\beta A \arrow[r, "\subseteq"] & \overline{R}_\gamma A \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & \overline{R}_\alpha A \arrow[r, "\subseteq"] & \overline{R}A \\
 & & \overline{R}_pA \arrow[ur, "\subseteq"'] & &
\end{tikzcd}$$

Remark 1.5 [2]: Let $\kappa=(X,R,\tau_K)$ be an approximation space and $A\subseteq X$. The implications between $Bnd(A)$ and $Bnd_j(A)$ are given in the following diagram for all $j\in\{p, s, \gamma, \alpha, \beta\}$.

$$\begin{tikzcd}
 & & Bnd_s(A) \arrow[dr, "\subseteq"] & & \\
Bnd_\beta(A) \arrow[r, "\subseteq"] & Bnd_\gamma(A) \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & Bnd_\alpha(A) \arrow[r, "\subseteq"] & Bnd(A) \\
 & & Bnd_p(A) \arrow[ur, "\subseteq"'] & &
\end{tikzcd}$$

2. EXACT AND ROUGH SETS

In the context of this section, we shall introduce new types of definability for a subset A of X in a topologized approximation space $\kappa = (X, R, \tau_K)$.

Definition 2.1[4]: Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and $A \subseteq X$. Then

(i) A is called totally R- definable (or exact) set if $\underline{R}A = A = \overline{R}A$.

(ii) A is called internally R- definable set if $A = \underline{R}A$.

(iii) A is called externally R- definable set if $A = \overline{R}A$.

(iv) A is called R- indefinable (or rough) set if $A \neq \underline{R}A$ and $A \neq \overline{R}A$.

Remark 2.1: Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and $A \subseteq X$.

- If A is totally R- definable set, then it is clopen in the topological space (X, τ_K).
- If A is internally R- definable set, then it is open in the topological space (X, τ_K).
- If A is externally R- definable set, then it is closed in the topological space (X, τ_K).
- If A is R- indefinable set, then it is neither open nor closed in the topological space (X, τ_K).

Definition 2.2: [3]. Let A be a subset of X in a topologized approximation space $\kappa = (X, R, \tau_K)$. Then

(i) A is called totally j-definable (briefly tot-j-def) (or j-exact) set if $\overline{R}_j A = A = \underline{R}_j A$,

(ii) A is called internally j-definable (briefly int-j-def) set if $A = \underline{R}_j A$,

(iii) A is called externally j-definable (briefly ext-j-def) set if $A = \overline{R}_j A$,

(iv) A is called j-indefinable (briefly j-indef) (or j-rough) set if $A \neq \underline{R}_j A$ and $A \neq \overline{R}_j A$, where $j \in \{ p, s, \gamma, \alpha, \beta \}$.

Definition 2.3: Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and $A \subseteq X$. Then

(i) A is called totally ij-definable (briefly tot-ij-def) (or ij-exact) set if $\underline{R}_i A = A = \overline{R}_j A$,

(ii) A is called internally ij-definable (briefly int-ij-def) set if $A = \underline{R}_i A$ and $A \neq \overline{R}_j A$,

(iii) A is called externally ij-definable (briefly ext-ij-def) set if $A \neq \underline{R}_i A$ and $A = \overline{R}_j A$,

(iv) A is called ij-indefinable (briefly ij-indef) (or ij-rough) set if $A \neq \underline{R}_i A$ and $A \neq \overline{R}_j A$, where $i, j \in \{ p, s, \gamma, \alpha, \beta \}$.

In Definition 2.3, if $i = j$, then $ij-\text{exact}$ (resp. $ij-\text{rough}$) set means $j-\text{exact}$ (resp. $j-\text{rough}$) set for all $i, j \in \{ p, s, \gamma, \alpha, \beta \}$.

Proposition 2.1: Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. The implications between j-exact and ij-exact subsets of X are given in the following diagram for all $i, j \in \{ p, s, \gamma, \alpha, \beta \}$.

$$
\begin{array}{ccccccc}
\beta\alpha-exact & \Rightarrow & \beta s(\text{or }\beta p)-exact & \Rightarrow & \beta\gamma-exact & \Rightarrow & \beta-exact \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\gamma\alpha-exact & \Rightarrow & \gamma s(\text{or }\gamma p)-exact & \Rightarrow & \gamma-exact & \Rightarrow & \gamma\beta-exact \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
s\alpha(\text{or }p\alpha)-exact & \Rightarrow & s(\text{or }p)-exact & \Rightarrow & s\gamma(\text{or }p\gamma)-exact & \Rightarrow & s\beta(\text{or }p\beta)-exact \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\alpha-exact & \Rightarrow & \alpha s(\text{or }\alpha p)-exact & \Rightarrow & \alpha\gamma-exact & \Rightarrow & \alpha\beta-exact
\end{array}
$$

Proof: We shall prove that αs -exact$\Rightarrow$ $\alpha\gamma$ -exact, and the other cases can be proved similarly. Let $A\subseteq X$ be αs -exact. Then $\underline{R}_\alpha A$ = A = $\overline{R}_s A$. Since $A\subseteq \overline{R}_\gamma A\subseteq \overline{R}_s A$ = A, then A = $\overline{R}_\gamma A$. Thus $\underline{R}_\alpha A$ = A = $\overline{R}_\gamma A$. Therefore A is $\alpha\gamma$ -exact.

The following example illustrates that the converse of Proposition 2.1 is not true in general.

Example 2.1: Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space such that X = {a, b, c, d} and R = {(a, a), (a, b), (b, a), (b, b), (d, d)}. Then
aR = {a, b} = bR ,cR = ϕ and dR = {d}. Hence
S = {{a, b}, {d}, ϕ},
$\mathcal{B}$ = {X, ϕ, {a, b}, {d}}. Thus
τ_K = {X, ϕ, {d}, {a, b}, {a, b, d}},
τ_K^* = {ϕ, X, {a, b, c}, {c, d}, {c}},
$SO(X)$ = {X, ϕ, {d}, {a, b}, {c, d}, {a, b, c}, {a, b, d}},
$PO(X)$ = {X, ϕ, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {a, b, d}, {a, c, d},{b, c, d}},
$\gamma O(X)$ = {X, ϕ, {a}, {b}, {d}, {a, b}, {a, d}, {b, d}, {c, d}, {a, b, c}, {a, b, d},{a, c, d}, {b, c, d}},
$\alpha O(X)$ = {X, ϕ, {d}, {a, b}, {a, b, d}},
$\beta O(X)$ = {X, ϕ, {a}, {b}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d},{a, b, c}, {a, b, d}, {b, c, d}, {a, c, d}}.
$SC(X)$ = {ϕ, X, {a, b, c}, {c, d}, {a, b}, {d}, {c}},
$PC(X)$ = {ϕ, X, {b, c, d}, {a, c, d}, {a, b, c}, {c, d}, {b, c}, {a, c}, {c}, {b}, {a}},
$\gamma C(X)$ = {ϕ, X, {b, c, d}, {a, c, d}, {a, b, c}, {c, d}, {b, c}, {a, c}, {a, b}, {d}, {c}, {b}, {a}},
$\alpha C(X)$ = {ϕ, X, {a, b, c}, {c, d}, {c}}, and
$\beta C(X)$ = {ϕ, X, {b, c, d}, {a, c, d}, {a, b, c}, {c, d}, {b, d}, {b, c}, {a, d}, {a, c}, {a, b}, {d}, {c}, {a}, {b}}.

If A = {a, d}, then A is a $\gamma\beta$ -exact set, since $\underline{R}_\gamma A$ = {a, d} = $\overline{R}_\beta A$. But A is not $s\beta$ -exact, since $\underline{R}_s A=\{d\}\neq A$.

Proposition 2.2: Let $\kappa=(X,R,\tau_K)$ be a topologized approximation space. The implications between internally ij-definable subsets of X are given in the following diagram for all $i, j\in$ { p , s, γ , α , β }.

$$
\begin{array}{ccccccc}
\text{int}-\beta\beta-\text{def} & \Rightarrow & \text{int}-\beta\gamma-\text{def} & \Rightarrow & \text{int}-\beta s(\text{or }\beta p)-\text{def} & \Rightarrow & \text{int}-\beta\alpha-\text{def} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\text{int}-\gamma\beta-\text{def} & \Rightarrow & \text{int}-\gamma\gamma-\text{def} & \Rightarrow & \text{int}-\gamma s(\text{or }\gamma p)-\text{def} & \Rightarrow & \text{int}-\gamma\alpha-\text{def} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\text{int}-s\beta(\text{or }p\beta)-\text{def} & \Rightarrow & \text{int}-s\gamma(\text{or }p\gamma)-\text{def} & \Rightarrow & \text{int}-ss(\text{or }pp)-\text{def} & \Rightarrow & \text{int}-s\alpha(\text{or }p\alpha)-\text{def} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\text{int}-\alpha\beta-\text{def} & \Rightarrow & \text{int}-\alpha\gamma-\text{def} & \Rightarrow & \text{int}-\alpha s(\text{or }\alpha p)-\text{def} & \Rightarrow & \text{int}-\alpha\alpha-\text{def}
\end{array}
$$

Proof: We shall prove that internally $\alpha\beta$ -definable $\Rightarrow$ internally $\alpha\gamma$ -definable and the other cases can be proved similarly. Let $A \subseteq X$ be internally $\alpha\beta$ -definable, then

$A = \underline{R}_\alpha A$ and $A \neq \overline{R}_\beta A$. Since $A \subset \overline{R}_\beta A \subseteq \overline{R}_\gamma A$, then $A \neq \overline{R}_\gamma A$. Thus $\underline{R}_\alpha A = A$ and $A \neq \overline{R}_\gamma A$. Therefore A is internally $\alpha\gamma$ -definable.

In the following example we illustrate that the converse of Proposition 2.2 is not true in general.

Example 2.2: Let $\kappa = (X, R, \tau_K)$ be the topologized approximation space which is given in Example 2.1. If $A = \{b, d\}$, then A is int-$\gamma\gamma$-def, since $\underline{R}_\gamma A = \{b, d\} = A$ and $\overline{R}_\gamma A = \{b, c, d\} \neq A$. But A is not int-$s\gamma$-def, since $\underline{R}_s A = \{d\} \neq A$.

Proposition 2.3: Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. The implications between externally ij-definable subsets of X are given in the following diagram for all $i, j \in \{p, s, \gamma, \alpha, \beta\}$.

$$\begin{array}{ccccccc}
\text{ext}-\beta\alpha-\text{def} & \Rightarrow & \text{ext}-\beta s(\text{or } \beta p)-\text{def} & \Rightarrow & \text{ext}-\beta\gamma-\text{def} & \Rightarrow & \text{ext}-\beta\beta-\text{def} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{ext}-\gamma\alpha-\text{def} & \Rightarrow & \text{ext}-\gamma s(\text{or } \gamma p)-\text{def} & \Rightarrow & \text{ext}-\gamma\gamma-\text{def} & \Rightarrow & \text{ext}-\gamma\beta-\text{def} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{ext}-s\alpha(\text{or } p\alpha)-\text{def} & \Rightarrow & \text{ext}-ss(\text{or } pp)-\text{def} & \Rightarrow & \text{ext}-s\gamma(\text{or } p\gamma)-\text{def} & \Rightarrow & \text{ext}-s\beta(\text{or } p\beta)-\text{def} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{ext}-\alpha\alpha-\text{def} & \Rightarrow & \text{ext}-\alpha s(\text{or } \alpha p)-\text{def} & \Rightarrow & \text{ext}-\alpha\gamma-\text{def} & \Rightarrow & \text{ext}-\alpha\beta-\text{def}
\end{array}$$

Proof: We shall prove that externally αs -definable $\Rightarrow$ externally $\alpha\gamma$ -definable and the other cases can be proved similarly. Let $A \subseteq X$ be externally αs -definable. Then

$A \neq \underline{R}_\alpha A$ and $A = \overline{R}_s A$. Since $A \subseteq \overline{R}_\gamma A \subseteq \overline{R}_s A = A$, then $A = \overline{R}_\gamma A$. Therefore A is externally $\alpha\gamma$ -definable.

In general the converse of Proposition 2.3 is not true. The following example illustrates this fact.

Example 2.3: Let $\kappa = (X, R, \tau_K)$ be the topologized approximation space which is given in Example 2.1. If $A = \{a, c\}$, then A is ext-$\gamma\gamma$-def, since $\underline{R}_\gamma A = \{a\} \neq A$ and $\overline{R}_\gamma A = \{a, c\} = A$. But A is not ext-γs-def, since $\overline{R}_s A = \{a, b, c\} \neq A$.

Proposition 2.4: Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space. The implications between j-rough and ij-rough subsets of X are given in the following diagram for all $i, j \in \{p, s, \gamma, \alpha, \beta\}$.

$$\begin{array}{ccccccc}
\beta-\text{rough} & \Rightarrow & \beta\gamma-\text{rough} & \Rightarrow & \beta s(\text{or } \beta p)-\text{rough} & \Rightarrow & \beta\alpha-\text{rough} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\gamma\beta-\text{rough} & \Rightarrow & \gamma-\text{rough} & \Rightarrow & \gamma s(\text{or } \gamma p)-\text{rough} & \Rightarrow & \gamma\alpha-\text{rough} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
s\beta(\text{or } p\beta)-\text{rough} & \Rightarrow & s\gamma(\text{or } p\gamma)-\text{rough} & \Rightarrow & s(\text{or } p)-\text{rough} & \Rightarrow & s\alpha(\text{or } p\alpha)-\text{rough} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\alpha\beta-\text{rough} & \Rightarrow & \alpha\gamma-\text{rough} & \Rightarrow & \alpha s(\text{or } \alpha p)-\text{rough} & \Rightarrow & \alpha-\text{rough}
\end{array}$$

Proof: We shall prove that $\alpha\gamma$ -rough $\Rightarrow \alpha s$ -rough and the other cases can be proved similarly. Let $A \subseteq X$ be $\alpha\gamma$ -rough. Then $A \neq \underline{R}_\alpha A$ and $A \neq \overline{R}_\gamma A$.

Since $A \subset \overline{R}_\gamma A \subseteq \overline{R}_s A$, then $A \neq \overline{R}_s A$. Thus $A \neq \underline{R}_\alpha A$ and $A \neq \overline{R}_s A$. Therefore A is αs -rough.

The following example illustrates that the converse of Proposition 2.4 is not true in general.

Example 2.4: Let $\kappa = (X, R, \tau_K)$ be the topologized approximation space which is given in Example 2.1. If $A = \{a, d\}$, then A is s -rough, since $\underline{R}_s A = \{d\} \neq A$ and $\overline{R}_s A = X \neq A$. But A is not γs -rough, since $\underline{R}_\gamma A = \{a, d\} = A$.

3. BOUNDARY REGIONS

The aim of this section is to introduce new levels of boundary regions in approximation spaces by using some classes of near open sets.

Definition 3.1: Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and $A \subseteq X$. The $i\,j$ -boundary region of A is denoted by $Bnd_{i\,j}(A)$ and it is defined by

$$Bnd_{i\,j}(A) = \overline{R}_i A - \underline{R}_j A, \text{ where } i,\ j \in \{\, p, s, \gamma, \alpha, \beta \,\}.$$

In Definition 3.1, if $i = j$, then $Bnd_{i\,j}(A) = Bnd_j(A)$ for all $i,\ j \in \{\, p, s, \gamma, \alpha, \beta \,\}$.

Proposition 3.1: Let $\kappa = (X, R, \tau_K)$ be a topologized approximation space and $A \subseteq X$. The implications between $Bnd_{i\,j}(A)$ are given in the following diagram for all $i, j \in \{\, p, s, \gamma, \alpha, \beta \,\}$.

$$\begin{tikzcd}
 & & Bnd_{is}(A) \arrow[dr, "\subseteq"] & \\
Bnd_{i\beta}(A) \arrow[r, "\subseteq"] & Bnd_{i\gamma}(A) \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & Bnd_{i\alpha}(A) \\
 & & Bnd_{ip}(A) \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

Proof: We shall prove that $Bnd_{\beta\gamma}(A) \subseteq Bnd_{\beta s}(A)$, and the other cases can be proved similarly. Since $\underline{R}_s A \subseteq \underline{R}_\gamma A$, then $\overline{R}_\beta A - \underline{R}_\gamma A \subseteq \overline{R}_\beta A - \underline{R}_s A$. Hence $Bnd_{\beta\gamma}(A) \subseteq Bnd_{\beta s}(A)$.

Example 3.1: Let $\kappa = (X, R, \tau_K)$ be the topologized approximation space which is given in Example 2.1. If $A = \{a, d\}$, then

$$Bnd_{\beta\gamma}(A) = \overline{R}_\beta A - \underline{R}_\gamma A = \{a, d\} - \{a, d\} = \phi, \text{ and}$$

$$Bnd_{\beta s}(A) = \overline{R}_\beta A - \underline{R}_s A = \{a, d\} - \{d\} = \{a\}.$$

Thus $Bnd_{\beta\gamma}(A) \subseteq Bnd_{\beta s}(A)$.

4. CONCLUSIONS

In this paper, we used different forms of near open and near closed sets to introduce new kinds of exact, internally definable, externally definable and rough sets.

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