International Research Journal of Pure Algebra-4(11), 2014, 631-635 Available online through www.rjpa.info

MINIMAL $\alpha r \omega$ -OPEN SETS AND MAXIMAL $\alpha r \omega$ -CLOSED SETS IN TOPOLOGICAL SPACES

R. S. Wali*1 and Prabhavati S. Mandalgeri²

¹Department of Mathematics, Bhandari Rathi College, Guledagudd-587 203, Karnataka State, India.

²Department of Mathematics, K. L. E'S, S.K. Arts College & H.S.K. Science Institute, Hubli-31, Karnataka State, India.

(Received On: 13-11-14; Revised & Accepted On: 24-11-14)

ABSTRACT

In this paper, a new class of sets called minimal $\alpha \omega$ -open sets and maximal $\alpha \omega$ -closed sets in topological spaces are introduced which are subclasses of $\alpha \omega$ -open sets and $\alpha \omega$ -closed sets respectively. We prove that the complement of minimal $\alpha \omega$ -closed set and some properties of the new concepts have been studied

Keywords: Minimal open set, Maximal closed set, Minimal ara-open set, Maximal ara-closed set.

1. INTRODUCTION

In the year 2001 and 2003, F. Nakaoka and N.oda [1] [2] [3] introduced and studied minimal open (resp. minimal closed) sets which are sub classes of open (resp. closed) sets. The complements of minimal open sets and maximal open sets are called maximal closed sets and minimal closed sets respectively. In the year 2014 R.S Wali & P.S Mandalgeri [4] introduced and studied $\alpha \omega$ -closed sets and $\alpha \omega$ -open sets in topological spaces.

Definition 1.1: [1] A proper non-empty open subset U of a topological space X is said to be minimal open set if any open set which is contained in U is ϕ or U.

Definition 1.2: [2] A proper non-empty open subset U of a topological space X is said to be maximal open set if any open set which contains U is either X or U.

Definition 1.3: [3] A proper non-empty closed subset F of a topological space X is said to be minimal closed set if any closed set which is contained in F is ϕ or F.

Definition 1.4: [3] A proper non-empty closed subset F of a topological space X is said to be maximal closed set if any closed set which contains F is either X or F.

Definition 1.5: [4] A subset A of (X, τ) is called $\alpha \tau \omega$ -closed set if $\alpha Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau \omega$ -open in X.

Definition 1.6: [4] A subset A in (X, τ) is called $\alpha r \omega$ -open set in X if A^c is $\alpha r \omega$ -closed set in X.

2. MINIMAL αrω-OPEN SETS

Definition 2.1: A proper non-empty $\alpha r \omega$ -open subset U of X is said to be minimal $\alpha r \omega$ -open set if any $\alpha r \omega$ -open set which is contained in U is ϕ or U.

Remark 2.2: Minimal open sets and Minimal $\alpha r \omega$ -open sets are independent of each other as seen from the following example.

*Corresponding author: R. S. Wali^{*1}, ¹Department of Mathematics, Bhandari Rathi College, Guledagudd-587 203, Karnataka State, India.

Example 2.3: Let X= {a, b, c, d} be with the topology $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$

Minimal open sets are = {{a}, {c, d}} $\alpha \omega$ -open sets are = {X, ϕ , {a}, {b}, {c}, {c, d}, {a, c, d}}

Minimal $\alpha r \omega$ -open sets are = {{a}, {b}, {c}}

Remark 2.4: From the Known results and by the above example we have the following implications.



Theorem 2.5:

(i) Let U be a minimal $\alpha r \omega$ -open set and W be a $\alpha r \omega$ -open set then $U \cap W = \phi$ or $U \subset W$. (ii) Let U and V be minimal $\alpha r \omega$ -open sets then $U \cap V = \phi$ or U = V.

Proof:

(i) Let U be a minimal $\alpha \omega$ -open set and W be a $\alpha \omega$ -open set. If $U \cap W = \phi$, then there is nothing to prove but if $U \cap W \neq \phi$ then we have to prove that $U \subset W$. Suppose $U \cap W \neq \phi$ then $U \cap W \subset U$ and $U \cap W$ is $\alpha \omega$ -open as the finite intersection of $\alpha \omega$ -open sets is a $\alpha \omega$ -open set. Since U is a minimal $\alpha \omega$ -open set, we have $U \cap W = U$ therefore $U \subset W$.

(ii) Let U and V be minimal $\alpha r \omega$ -open sets. Suppose $U \cap V \neq \phi$ then we see that $U \subset V$ and $V \subset U$ by (i) Therefore U=V.

Theorem 2.6: Let U be a minimal $\alpha r \omega$ -open set. If x is an element of U then U \subset W for any open neighbourhood W of x.

Proof: Let U be a minimal αr_{ω} -open set and x be an element of U. Suppose there exits an open neighbourhood W of x such that $U \not\subset W$ then $U \cap W$ is a αr_{ω} -open set such that $U \cap W \subset U$ and $U \cap W \neq \phi$. Since U is a minimal αr_{ω} -open set, we have $U \cap W = U$ that is $U \subset W$. This contradicts our assumption that $U \not\subset W$. Therefore $U \subset W$ for any open neighbourhood W of x.

Theorem 2.7: Let U be a minimal $\alpha r \omega$ -open set, if x is an element of U then U \subset W for any $\alpha r \omega$ -open set W containing x.

Proof: Let U be a minimal $\alpha r \omega$ -open set containing an element x. Suppose there exists an $\alpha r \omega$ - open set W containing x such that $U \not\subset W$ then $U \cap W$ is an $\alpha r \omega$ -open set such that $U \cap W \subset U$ and $U \cap W \neq \phi$ Since U is a minimal $\alpha r \omega$ -open set, we have $U \cap W = U$ that is $U \subset W$. This contradicts our assumption that $U \not\subset W$. Therefore $U \subset W$ for any $\alpha r \omega$ -open set W containing x.

Theorem 2.8: Let U be a minimal $\alpha r \omega$ -open set then U= $\cap \{W: W \text{ is any } \alpha r \omega$ -open set containing x} for any element x of U

Proof: By theorem 2.7 and from the fact that U is a $\alpha r \omega$ -open set Containing x, We have $U \subset \cap \{W: W \text{ is any } \alpha r \omega$ -open set containing x} $\subset U$. Therefore we have the result.

Theorem 2.9: Let U be a non-empty αrω-open set then the following three conditions are equivalent.

(i) U is a minimal αrω-open set

(ii) $U \subset \alpha r \omega$ -cl(S) for any non-empty subset S of U.

(iii) $\alpha r \omega$ -cl(U)= $\alpha r \omega$ -cl(S) for any non-empty subset S of U.

Proof:

(i) => (ii): Let U be a minimal $\alpha \omega$ -open set and S be a non-empty subset of U. Let $x \in U$ by theorem 2.7 for any $\alpha \omega$ -open set W containing x, $S \subset U \subset W$ which implies $S \subset W$. Now $S = S \cap U \subset S \cap W$. Since S is non-empty therefore $S \cap W \neq \Phi$. Since W is any $\alpha \omega$ -open set containing x by one of the theorem, we know that, for an $x \in X$, $x \in \alpha \omega$ -cl(A) iff $V \cap A \neq \Phi$ for any every $\alpha \omega$ -open set V Containing x, that is $x \in U$ implies $x \in \alpha \omega$ -cl(S) which implies $U \subset \alpha \omega$ -cl(S) for any non-empty subset S of U.

(ii) => (iii): Let S be a non-empty subset of U, that is $S \subset U$ which implies $\alpha r \omega - cl(S) \subset \alpha r \omega - cl(U) - (a)$ Again from (ii) $U \subset \alpha r \omega - cl(S)$ for any non-empty subset S of U. Which implies $\alpha r \omega - cl(U) \subset \alpha r \omega - cl(\alpha r \omega - cl(S)) = \alpha r \omega - cl(S)$ i.e., $\alpha r \omega - cl(U) \subset \alpha r \omega - cl(S) - (b)$, from (a) and (b), $\alpha r \omega - cl(U) = \alpha r \omega - cl(S)$ for any non empty subset S of U.

(iii) => (i): From (iii) we have $\alpha \tau \omega$ -cl(U) = $\alpha \tau \omega$ -cl(S) for any non-empty subset S of U. Suppose U is not a minimal $\alpha \tau \omega$ -open set then there exist a non-empty $\alpha \tau \omega$ -open set V such that $V \subset U$ and $V \neq U$. Now there exists an element $a \in U$ such that $a \notin V$ which implies $a \in V^c$ that is $\alpha \tau \omega$ -cl({a}) \subset \alpha \tau \omega-cl{(V^c)} = V^c, as V^c is a $\alpha \tau \omega$ -closed set in X. It follows that $\alpha \tau \omega$ -cl({a}) $\neq \alpha \tau \omega$ -cl(U). This is contradiction to fact that $\alpha \tau \omega$ -cl({a})= $\alpha \tau \omega$ -cl(U) for any non empty subset {a} of U. Therefore U is a minimal $\alpha \tau \omega$ -open set.

Theorem 2.10: Let V be a non-empty finite $\alpha r \omega$ -open set, then there exists at least one (finite) minimal $\alpha r \omega$ -open set U such that U \subset V.

Proof: Let V be a non-empty finite $\alpha r \omega$ -open set. If V is a minimal $\alpha r \omega$ -open set, we may set U=V. If V is not a minimal $\alpha r \omega$ -open set, then there exists a (finite) $\alpha r \omega$ -open set V₁ such that $\phi \neq V_1 \subset V$. If V₁ is a minimal $\alpha r \omega$ -open set, we may set U=V₁. If V₁ is not a minimal $\alpha r \omega$ -open set then there exists a (finite) $\alpha r \omega$ -open set V₂ such that $\phi \neq V_2 \subset V_1$. Continuing this process we have a sequence of $\alpha r \omega$ -open set V_k ... $\subset V_3 \subset V_2 \subset V_1 \subset V$. Since V is a finite set, this process repeats only finitely then finally we get a minimal $\alpha r \omega$ -open set U=V_n for some positive integer n.

Corollary 2.11: Let X be a locally finite space and V be a non-empty $\alpha r \omega$ -open set then there exists at least one(finite) minimal $\alpha r \omega$ -open set such that U \subset V.

Proof: Let X be a locally finite space and V be a non empty $\alpha r \omega$ -open set. Let $x \in V$ since X is a locally finite space we have a finite open set Vx such that $x \in V_x$ then $V \cap V_x$ is a finite $\alpha r \omega$ -open set. By theorem 2.10 there exist at least one (finite) minimal $\alpha r \omega$ -open set U such that $U \subset V \cap V_x$ that is $U \subset V \cap V_x \subset V$. Hence there exists at least one (finite) minimal $\alpha r \omega$ -open set U such that $U \subset V \cap V_x$ that is $U \subset V \cap V_x \subset V$.

Corollary 2.12: Let V be a finite minimal open set then there exist at least one(finite) minimal $\alpha r \omega$ -open set U such that U \subset V.

Proof: Let V be a finite minimal open set then V is a non-empty finite $\alpha r \omega$ -open set, by theorem 2.10 there exist at least one (finite)minimal $\alpha r \omega$ -open set U such that U \subset V.

Theorem 2.13: Let U and U_{λ} be minimal $\alpha r \omega$ -open sets for any element λ of Λ . If $U \subset \bigcup_{\lambda \in \Lambda} U_{\lambda}$ then there exists $\lambda \in \Lambda$ an element $\lambda \in \Lambda$ such that $U = U_{\lambda}$.

Proof: Let $U \subset \bigcup_{\lambda \in \Lambda} U_{\lambda}$ then $U \cap \bigcup_{\lambda \in \Lambda} U_{\lambda} = U$ that is $\bigcup_{\lambda \in \Lambda} (U \cap U_{\lambda}) = U$, also by Theorem 2.5 (ii) $U \cap U_{\lambda} = \phi$ for any $\lambda \in \Lambda$ It follows that there exist an element $\lambda \in \Lambda$ such that $U = U_{\lambda}$.

Theorem 2.14: Let U and U_{λ} be minimal $\alpha r \omega$ -open sets for any element $\lambda \in \Lambda$. If $U = U_{\lambda}$ for any element λ of Λ then $\bigcup_{\lambda \in \Lambda} U_{\lambda} \cap U = \varphi$

Proof: Suppose that $\bigcup_{\lambda \in \Lambda} U_{\lambda} \cap U \neq \phi$ that is $\bigcup_{\lambda \in \Lambda} (U_{\lambda} \cap U) \neq \phi$. then there exists an element $\lambda \in \Lambda$ such that $U \cap U_{\lambda} \neq \phi$ by theorem 2.5 (ii) we have $U = U_{\lambda}$, which contradicts the fact that $U \neq U_{\lambda}$ for any $\lambda \in \Lambda$ then $\bigcup U_{\lambda} \cap U = \phi$.

Theorem 2.15: Let U_{λ} be a minimal $\alpha r \omega$ -open set for any element $\lambda \in \Lambda$ and $U_{\lambda} \neq U_{\mu}$ for any element λ and μ of Λ with $\lambda \neq \mu$ assume that $|\Lambda| \ge 2$. Let μ be any element of Λ then $\bigcup U_{\lambda} \cap U_{\mu} = \varphi$.

Proof: Put $U=U_{\mu}$ in theorem 2.14, then we have the result.

Corollary 2.16: Let U_{λ} be a minimal $\alpha r \omega$ -open set for any element $\lambda \in \Lambda$ and $U_{\lambda} \neq U_{\mu}$ for any element λ and μ of Λ with $\lambda \neq \mu$. If Γ a proper non-empty subset of Λ then $\bigcup_{\lambda \in \Lambda - \Gamma} U_{\lambda} \cap \bigcup_{\gamma \in \Gamma} U_{\gamma} = \varphi$

Theorem 2.17: Let U_{λ} and U_{γ} be minimal $\alpha r \omega$ -open sets for any element $\lambda \in \Lambda$ and $\gamma \in \Gamma$ If there exists an element γ of Γ such that $U_{\lambda} \neq U_{\gamma}$ for any element λ of Λ , then $\bigcup U\gamma \not\subset \bigcup U_{\lambda}$

Proof: Suppose that an element γ^1 of Γ satisfies $U_{\lambda} = U_{\gamma 1}$ for any element λ of Λ . if $\bigcup_{\gamma \in \Gamma} U\gamma \subset \bigcup_{\lambda \in \Lambda} U_{\lambda}$, then we see

 $U_{\gamma 1} \subset \bigcup_{\lambda \in \Lambda} U_{\lambda}$ by theorem 2.13 there exists an element λ of Λ such that $U_{\gamma 1} = U_{\lambda}$ which is a contradiction. It follows that $\bigcup_{\gamma \in \Gamma} U_{\gamma} \not\subset \bigcup_{\lambda \in \Lambda} U_{\lambda}$

3. MAXIMAL αrω-closed SETS

Definition 3.1: A proper non-empty $\alpha r \omega$ -closed subset F of X is said to be Maximal $\alpha r \omega$ -closed set if any $\alpha r \omega$ -closed set which contains F is either X or F.

Remark 3.2: Maximal closed sets and Maximal $\alpha r \omega$ -closed sets are independent each other as seen from the following implication.

Example 3.3: Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$ be a topological space.

Closed sets are ={X, \emptyset , {b}, {a, b}, {b, c, d}}Maximal closed sets are = {{a, b}, {b, c, d}} ar ω -closed sets are ={X, ϕ , {b}, {a, b}, {b, c, d}, {a, c, d}, {a, b, d}}

Maximal $\alpha r \omega$ -closed sets are = { {b, c, d}, {a, c, d}, {a, b, d} }

Remark 3.4: From the known results and by the above example 3.3 we have the following implication.



Theorem 3.5: A proper non-empty subset F of X is Maximal $\alpha r \omega$ -closed set iff X-F is a minimal $\alpha r \omega$ -open set.

Proof: Let F be a Maximal $\alpha r \omega$ -closed set, suppose X-F is not a minimal $\alpha r \omega$ -open set, then there exists a $\alpha r \omega$ -open set $U \neq X - F$ such that $\phi \neq U \subset X - F$ that is $F \subset X - U$ and X-U is a $\alpha r \omega$ -closed set. This contradicts our assumption that F is a minimal $\alpha r \omega$ - open set.

Conversely, let X-F be a minimal $\alpha \omega$ -open set. Suppose F is not a Maximal $\alpha \omega$ -closed set then there exist a $\alpha \omega$ -closed set $E \neq F$ such that $F \subset E \neq X$ that is $\varphi \neq X \cdot E \subset X \cdot F$ and X-E is a $\alpha \omega$ -open set. This contradicts our assumption that X-F is a minimal $\alpha \omega$ -open set. Therefore F is a Maximal $\alpha \omega$ -closed set.

Theorem 3.6:

(i) Let F be a Maximal $\alpha r \omega$ -closed set and W be a $\alpha r \omega$ -closed set Then FUW=X or WCF. (ii) Let F and S be Maximal $\alpha r \omega$ -closed sets then FUS=X or F=S

Proof:

(i): Let F be a Maximal $\alpha r \omega$ -closed set and W be a $\alpha r \omega$ -closed set if FUW=X then there is nothing to prove but if FUW \neq X, then we have to prove that W \subset F. Suppose FUW \neq X then F \subset FUW and FUW is $\alpha r \omega$ -closed as the finite union of $\alpha r \omega$ -closed set is a $\alpha r \omega$ -closed set we have FUW=X or FUW=F. Therefore FUW=F which implies W \subset F.

(ii): Let F and S be Maximal $\alpha r \omega$ -closed sets. Suppose $F \cup S \neq X$ then we see that $F \subset S$ and $S \subset F$ by (i) therefore F = S.

Theorem 3.7: Let F be a Maximal $\alpha r \omega$ -closed set. If x is an element of F then for any $\alpha r \omega$ -closed set S containing x, FUS=X or S⊂F

Proof: Proof is similar to 2.7 theorems.

Theorem 3.8: Let F_{α} , F_{β} , F_{γ} be Maximal $\alpha r \omega$ -closed sets such that $F_{\alpha} \neq F_{\beta}$ if $F_{\alpha} \cap F_{\beta} \subset F_{\gamma}$, then either $F_{\alpha} = F_{\gamma}$ or $F_{\beta} = F_{\gamma}$.

Proof: Given that $F_{\alpha} \cap F_{\beta} \subset F_{\gamma}$, if $F_{\alpha} = F_{\gamma}$ then there is nothing to prove but if $F_{\alpha} \neq F_{\gamma}$ then We have to prove $F_{\beta} = F_{\gamma}$.

Now we have $F_{\beta} \cap F_{\gamma} = F_{\beta} \cap (F_{\gamma} \cap X)$ $= F_{\beta} \cap (F_{\gamma} \cap (F_{\alpha} \cup F_{\beta})$ (by theorem 3.6 (ii)) $= F_{\beta} \cap ((F_{\gamma} \cap F_{\alpha}) \cup (F_{\gamma} \cap F_{\beta}))$ $= (F_{\beta} \cap F_{\gamma} \cap F_{\alpha}) \cup (F_{\beta} \cap F_{\gamma} \cap F_{\beta})$ $= (F_{\alpha} \cap F_{\beta}) \cup (F_{\gamma} \cap F_{\beta})$ (by $F_{\beta} \cap F_{\gamma} \subset F_{\beta}$) $= (F_{\alpha} \cup F_{\gamma}) \cap F_{\beta}$ $= X \cap F_{\beta}$ (since F_{α} , and F_{γ} are Maximal $\alpha r \omega$ -closed sets by thm 3.6 (ii) $F_{\alpha} \cup F_{\gamma} = X$) $= F_{\beta}$

That is $F_{\beta} \cap F_{\gamma} = F_{\beta}$ implies $F_{\beta} \subset F_{\gamma}$, since F_{β} , F_{γ} are maximal $\alpha r \omega$ -closed sets , we have $F_{\beta} = F_{\gamma}$.

Theorem 3.9: Let F_{α} , F_{β} , F_{γ} be Maximal $\alpha r \omega$ -closed sets which are different from each other then $(F_{\alpha} \cap F_{\beta}) \not = (F_{\alpha} \cap F_{\gamma})$.

Proof: Let $(F_{\alpha} \cap F_{\beta}) \notin (F_{\alpha} \cap F_{\gamma})$ which implies $(F_{\alpha} \cap F_{\beta}) \cup (F\gamma \cap F_{\beta}) \subset (F_{\alpha} \cap F_{\gamma}) \cup (F\gamma \cap F_{\beta})$ which implies $(F_{\alpha} \cup F_{\gamma}) \cap F_{\beta} \subset F_{\gamma} \cap (F_{\alpha} \cup F_{\beta})$ since by theorem 3.6 (ii) $F_{\alpha} \cap F_{\gamma} = X$ and $F_{\alpha} \cap F_{\beta} = X$ which implies $X \cap F_{\beta} \subset F_{\gamma} \cap X$ which implies $F_{\beta} \subset F_{\gamma}$. From the definition of Maximal $\alpha \omega$ -closed set it follows that $F_{\beta} = F_{\gamma}$.

This is contradiction to the fact that Let F_{α} , F_{β} , F_{γ} are different from each other. Therefore $(F_{\alpha} \cap F_{\beta}) \not \subset (F_{\alpha} \cap F_{\gamma})$.

Theorem 3.10: Let F be a Maximal $\alpha r \omega$ -closed set and x be an element of F then E S: S is a $\alpha r \omega$ -closed set containing x such that FUS \neq X}.

Proof: By theorem 3.7 and from the fact that F is a $\alpha r \omega$ -closed set containing x we have $F \subset \bigcup \{S: S \text{ is a } \alpha r \omega \text{-closed set containing x such that } F \cup S \neq X \} \subset F$ therefore we have the result.

Theorem 3.11: Let F be a Proper non-empty co-finite $\alpha r \omega$ -closed subset then there exists (co-finite) Maximal $\alpha r \omega$ -closed set E such that $F \subset E$.

Proof: Let F be a non-empty co-finite $\alpha \omega$ -closed set. If F is a Maximal $\alpha \omega$ -closed set, we may set E=F. If F is not a Maximal $\alpha \omega$ -closed set, then there exists a (co-finite) $\alpha \omega$ -closed set F₁ such that $F \subseteq F_1 \neq X$. If F₁ is a Maximal $\alpha \omega$ -closed set, we may set E = F₁. If F₁ is not a Maximal $\alpha \omega$ -closed set, then there exists a (co-finite) $\alpha \omega$ -closed set set F₂ such that $F \subseteq F_1 \subseteq F_2 \neq X$ continuing this process we have a sequence of $\alpha \omega$ -closed sets $F \subseteq F_1 \subseteq F_2 \subseteq ... \subseteq F_k \subseteq ...$ since F is a co-finite set, this process repeats only finitely then finally we get a minimal $\alpha \omega$ -open set E=F_n for some positive integer n.

Theorem 3.12: Let F be a Maximal $\alpha r \omega$ -closed set. If x is an element of X-F then X-F c for any $\alpha r \omega$ -closed set containing set E containing x

Proof: Let F be a Maximal $\alpha \omega$ -closed set and $x \in X-F$. E F for any $\alpha \omega$ -closed set E containing x then EUF=X by theorem 3.6(ii). Therefore $X-F \subset E$

ACKNOWLEDGMENT

The Authors would like to thank the referees for useful comments and suggestions.

REFERENCES

- 1. F.Nakaoka and F.Oda, Some Application of minimal open sets, Int. J. Math. Math. sci. Vol 27, No.8, 471-476(2001).
- 2. F.Nakaoka and F.Oda, Some Properties of Maximal open sets, Int. J. Math. Math. sci. Vol 21, No.21, 1331-1340(2001).
- 3. F.Nakaoka and F.Oda, on minimal closed sets, Proceeding of topological spaces and It's application, 5, 19-21(2003).
- 4. R.S.wali and P.S.Mandalgri, On α Regular ω -closed sets in topological spaces, Int. J. of Math. Archive 5(10), 2014, 68-76.

Source of Support: Nil, Conflict of interest: None Declared

[Copy right © 2014 This is an Open Access article distributed under the terms of the International Research Journal of Pure Algebra (IRJPA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]