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THE POLYGONAL PROPERTIES OF A TOPOLOGICAL SPACE GRAPH

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ABSTRACT

In this paper we adopt a novel topological approach to graphs, focusing especially on Topological space graphs we also consider topological models for graphs and studied polygonal properties of this graph. Also we obtained some results on girth, eccentricity and the number of triangles in this graph.

***Key words**: Graph, Topological space graph, girth, eccentricity.*

***Classification:** 68R10, 90C35, 92E10.*

1. INTRODUCTION

Most of the work in function space topologies concerns continuous functions. As soon as we begin to consider function spaces of noncontinuous functions we come face to face with some extremely difficult problems. An almost continuous function is one whose graph can be approximated by graphs of continuous functions . The need to introduce a suitable topology for the function space of almost continuous functions arose when the author was investigating the essential fixed points of such functions in this paper.

Given a topological space τ on a non -empty set X, we define a graph $G_X(\tau)$ whose vertex set T and arcs exist between two elements of τ if one is included in another, which is introduced by our self. To this end, we first introduce or recall some basic definitions and notations, and collect some related simple facts. Then, we prove our main results on polygonal properties of this graph. Mainly, girth, eccentricity and the number of triangles in this graph.

Given a non-empty set X and a topology τ on X, we define a graph whose vertices are members of τ and arcs are defined among vertices if one node is included in another one which is known as “Topological space graph”. This construction is of interest in the context of polygonal properties, (like girth , eccentricity and triangles) of a graph analysis.

2. PRELIMINARIES

2.1 Definition: A graph G is a triple consisting of a vertex set V(G), an edge set E(G), and a relation that associates with each edge two vertices (not necessarily distinct) called its end points.

2.2 Example (Königsberg bridge problem): The city of Königsberg (now Kaliningrad) used to have seven bridges across the river, linking the banks with two islands. The people living in Königsberg had a game where they would try to walk across each bridge once and only once. You can chose where to start.

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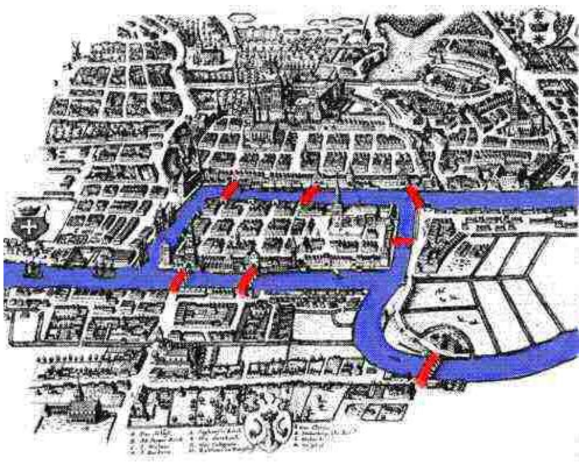
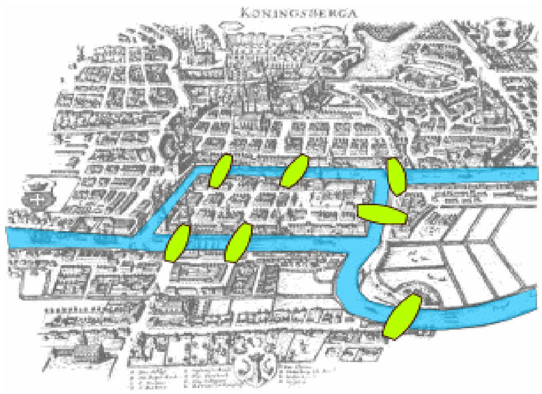
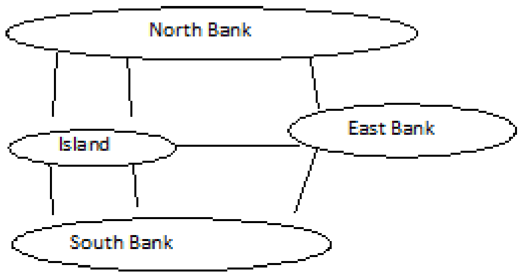


Solution: Back in the 18th century in the Prussian city of Königsberg, a river ran through the city and seven bridges crossed the forks of the river. The river and the bridges are highlighted in the picture to the right[1].

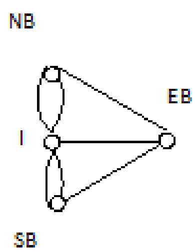


As a weekend amusement, townsfolk would see if they could find a route that would take them across every bridge once and return them to where they started.

Leonard Euler (pronounced OY-lur), one of the most prolific mathematicians ever, looked at this problem in 1735, laying the foundation for graph theory as a field in mathematics. To analyze this problem, Euler introduced edges representing the bridges:



Since the size of each land mass it is not relevant to the question of bridge crossings, each can be shrunk down to a vertex representing the location:



Notice that in this graph there are *two* edges connecting the north bank and island, corresponding to the two bridges in the original drawing. Depending upon the interpretation of edges and vertices appropriate to a scenario, it is entirely possible and reasonable to have more than one edge connecting two vertices.

2.3 Definitions:

(i) A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.
(ii) A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle
(iii) A graph G is **connected** if each pair of vertices in G belongs to a path; otherwise, G is disconnected.
(iv) A graph is **finite** if its vertex set and edge set are finite.

We adopt the convention that every graph mentioned in this paper is finite, unless explicitly constructed otherwise.

2.4 Definition: A loop is an edge whose endpoints are equal. Multiple edges are edges having the same pair of endpoints

2.5 Definition: The girth of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth.

2.6 Definition: A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.

2.7 Definitions: (i) Let X be a non-empty set, τ be an arbitrary topology on X. A graph $G_X(\tau)$ whose vertex set is τ and two vertices u, v of $G_X(\tau)$ are adjacent if and only if $u \cup v = u \text{ or } v$. This graph is called "**Topological space graph"** .It is denoted by $G_X(\tau)$.

We refer the reader to for basic definitions and facts concerning topological spaces and graph theory. We write X for the non-empty set and, given a topological space T and two distinct elements $u, v \in \tau$, we denote

- by $G_X(\tau)$, the topological space graph on a topological space (X, τ).
- by τ the vertex set of the topological space graph $G_X(\tau)$.

The following lemma collects some obvious observations whose verification (partly indicated in the lemma) is left to the reader:

2.8 Lemma:
(i) Let u, v be any two subsets of a universal set X. Then $u \cup v = u \text{ or } v \Leftrightarrow u \cap v = v \text{ or } u$.
(ii) Let u, v be any two subsets of a universal set X. Then $u \subseteq v \Leftrightarrow u \cup v = v \Leftrightarrow u \cap v = u$.
(iii) Let u be any subset of a universal set X. Then $\emptyset \subseteq u \ \& \ u \subseteq X$.

3. THE RESULTS ON TOPOLOGICAL SPACE GRAPH $G_X(T)$

In this section we mainly studied and obtained some results on polygonal properties of topological space graphs.

A generalized **Moore graph** is a regular graph of degree r where the counts of vertices at each distance , 1, 2,….. from *any* vertex are 1, r, r(r – 1), ..., with the last distance count not necessarily filled up.

3.1 Result: Let τ be a non-trivial topology. The girth of a topological space graph $G_X(\tau)$ is three.

Proof: Easily we see that $G_X(\tau)$ is a simple graph.

Let $\phi, X, v \in V(G_X(\tau))$
$\phi \cup X = X$, $v \cup X, = X$, $\phi \cup v = v$
$\Rightarrow <\phi, X>, <v, X>, <\phi, v>$ are edges in $G_X(\tau)$
$\Rightarrow <\phi, X, v, \phi>$ is a triangle in $G_X(\tau)$

$\Rightarrow < \phi, X, v, \phi >$ is a smallest cycle in $G_X(\tau)$

Thus, $G_X(\tau)$ has a girth of length three.

3.2. Result: Let τ be a non-trivial topology. Every topological space graph $G_X(\tau)$ is moore graph if $dia(G_X(\tau)) = 1$

Proof: $dia(G_X(\tau)) = 1 \Rightarrow \sup\{d(u,v) \mid \forall u, v \in V(G_X(\tau))\} = 1 \Rightarrow$ every pair of nodes in $G_X(\tau)$ has an edge $\Rightarrow G_x(\tau)$ is a complete graph. $\Rightarrow$ girth of length is three because the smallest cyclic in any complete graph is three . $\Rightarrow$ girth of tenth in $G_X(\tau)$ is 3=2.1+1 $\Rightarrow G_x(\tau)$ is a moore graph.

3.3 Result: Let τ be a non-trivial topology .show that topological space graph $G_X(\tau)$ is never generalized polygon.

Proof: Sup pose $G_X(\tau)$ is a generalized polygon $\Rightarrow$ girth of $G_X(\tau)$ is $2\ d$, where d be the $dia(G_X(\tau))$ $\Rightarrow$ since $G_X(\tau)$ have diameter 1 or 2, so girth of $G_X(\tau)$ is 2x1 or 2x2 $\Rightarrow$ girth of topological space graph $G_X(\tau)$ is either 2 or 4 $\Rightarrow$ Which is impossible because girth of topological space graph $G_X(\tau)$ is there.

Thus, $G_X(\tau)$ is never generalized polygon graph.

3.4 Result: The number of girth in any topological graph $G_X(\tau)$ is equal to the number of triangles in $G_X(\tau)$

Proof: Let A be the set of girth in $G_X(\tau)$ & B be the set of triangles in $G_X(\tau)$. Now, $< v_1, v_2, v_3, v_1 > \in B \Leftrightarrow < v_1, v_2, v_3, v_1 >$ is a triangle in $G_X(\tau) \Leftrightarrow < v_1, v_2, v_3, v_1 >$ is a cycle in $G_X(\tau)$ whose length is three $\Leftrightarrow < v_1, v_2, v_3, v_1 > \in A$

Thus, the number of girth in any topological space graph $G_X(\tau)$ is equal to the number of triangles in $G_X(\tau)$.

3.5 Result: Let τ be a non-trivial topology on a non-empty set X. show that independent number of $G_X(\tau)$ is atmost two times of the number of pairs in $G_X(\tau)$ whose distance is two.

Proof: Let 'n' be the independent number in $G_X(\tau)$. So, there exist a largest independent set in $G_X(\tau)$ whose order is n . Let $u, v_1 \in V(G_X(\tau)) \Rightarrow u \cup v, \neq u\, or\, v_1 \Rightarrow < u, v_1 >$ is not an edge in $G_X(\tau)$

$$\Rightarrow d(u, v_1) = 2 \ (\because < u, \varphi >, < v_1, \varphi >) \quad (1)$$

Similarly, $u, v_2 \in V(G_X(\tau)) \Rightarrow u \cup v_2 \neq u\, or\, v_2 \Rightarrow < u, v_2 >$ is not an edge in $G_X(\tau)$

$$\Rightarrow d(u, v_2) = 2(\because < u, \phi >, < \phi, v_2 >) \quad (2)$$

Thus from equations (1),(2) it is clear that the size of the largest independent set in $G_X(\tau)$ is atmost 2 times of the set of order pairs whose distance is two.

3.6 Result: In any topological space graph $G_X(\tau)$, the radius is one

Proof: $Rad(G_{X(\tau)}) = \underset{v_i \in V(G_X(\tau))}{Min} ecc(v_i) = \text{Min}\{ecc(\phi), \ldots, ecc(X)\} (\because \phi, X \in G_X(\tau))$

$$= \text{Min}\{ \underset{v_2 \in V(G_X(\tau))}{Max} d_{(G_X(\tau))}(\phi, v_2), \ldots., \underset{v_2 \in V(G_X(\tau))}{Max} d_{(G_X(\tau))}(X, v_2)\} = Min\{1, \ldots, 1\} = 1$$

3.7 Result: In any topological space graph $G_X(\tau)$, the eccentricity of any node is either one or two.

Proof: Let $v_1 \in \tau$

Case-1: suppose that $v_1 \cup v_2 = v_2 \, or \, v_1$ for every $v_2 \in V(G_X(\tau))$

$\Rightarrow d_{G_X(\tau)}(v_1, v_2) = 1, \forall v_2 \in V(G_X(\tau)) \Rightarrow \underset{v_2 \in G_X(\tau)}{Max} d_{G_X(\tau)}(v_1, v_2) = 1 \Rightarrow ecc(v_1) = 1$

Case-2: suppose that $v_1 \cup v_2 \neq v_2 \, or \, v_1$ for every $v_2 \in V(G_X(\tau))$.

$\Rightarrow$ there exist at least one $v_3 \in G_X(\tau)$ such that $v_1 \cup v_3 \neq v_1 or \, v_3$

$\Rightarrow d_{G_X(\tau)}(v_1, v_3) = 2 \text{ or } 1 \Rightarrow \underset{v_2 \in V(G_X(\tau))}{Max} d_{G_X(\tau)}(v_1, v_2) = 2 \; \left(\because d_{G_X(\tau)}(v_1, v_2) = 1 \text{ or } 2\right) \Rightarrow ecc(v_1) = 2$

Since $v_1 \in \tau$ arbitrary, so, the eccentricity of any node is either one or two in $G_X(\tau)$.

3.8 Result: Show that the eccentricity of ϕ, X in any $G_X(\tau)$ one

Proof: First show that eccentricity of ϕ is one

Now $ecc(\phi) = \underset{v_1 \in V(G_X(\tau))}{Max} d_{G_X(\tau)}(\phi, v_1) = \underset{v_1 \in \tau}{Max}\{1, ..., 1\} \quad (\because \phi \cup v_1 = v_1, \forall v_1 \in \tau) = 1$.

Next to show that the eccentricity of X is one.

$ecc(X) = \underset{v_1 \in \tau}{Max} d_{G_X(\tau)}(X, v_1) = \underset{v_1 \in \tau}{Max}\{1, ..., 1\} (\because X \cup v_1 = X, \forall v_1 \in \tau) = 1$

Thus the eccentrically of ϕ, X in any $G_X(\tau)$ is one.

3.9 Result: If $G_X(\tau)$ is not a complete graph, then all non-adjacent nodes in $G_X(\tau)$ of eccentricity is exactly two.

Proof: $G_X(\tau)$ is not a complete graph $\Rightarrow$ there exits at least one pair of nodes u, v in $G_x(\tau)$ such that

$u \cup v \neq u \, or \, v \;\; \Rightarrow d_{G_X(\tau)}(u, v) = 2 \;\; (\because u \cup \phi = u, v \cup \phi = v \Rightarrow < u, \phi >, < \phi, v >)$

$\Rightarrow \underset{v \in V(G_X(\tau))}{Max} d_{G_X(\tau)}(u, v) = 2 \qquad \left(\because d_{G_X(\tau)}(u, v) \leq 2\right) \;\; \Rightarrow ecc(u) = 2$

Similarly, $\Rightarrow d_{G_X(\tau)}(u, v) = 2 \;\; (\because u \cup \phi = u, v \cup \phi = v \Rightarrow < u, \phi >, < \phi, v >) \Rightarrow \underset{v \in V(G_X(\tau))}{Max} d_{G_X(\tau)}(u, v) = 2$

$\left(\because d_{G_X(\tau)}(u, v) \leq 2\right) \Rightarrow ecc(v) = 2$

Thus, it is clear that all non-adjacent nodes in $G_X(\tau)$ has an eccentricity two.

3.10 Result: $G_X(\tau)$ is a complete graph if and only if eccentricity of all nodes is equal to the radius of the graph $G_X(\tau)$

Proof: Let $G_X(\tau)$ is a complete graph

$\Rightarrow d_{G_X(\tau)}(v_1, u) = 1, \forall v_1 \in \tau \Rightarrow \underset{u \in \tau}{Max} d_G(v_1, u) = 1, \forall v_1 \in \tau$

$\Rightarrow ecc(v_1) = 1, \forall v_1 \in \tau$

$\Rightarrow ecc(v_1) = radG_x(\tau), \forall v_1 \in \tau \ \left(\therefore rad\left(G_x(\tau)\right) = 1\right)$

Conversely,

Let $ecc(v_1) = rad\left(G_x(\tau)\right), \forall v_1 \in \tau$

$\Rightarrow ecc(v_1) = 1, \forall v_1 \in \tau \ \left(\because rad\left(G_x(\tau)\right) = 1\right)$

$\Rightarrow \max_{v_2 \in \tau} d_G(v_1, v_2) = 1, \forall v_1 \in \tau$

$\Rightarrow < v_1, v_2 > \left(G_x(\tau)\right) \forall v_1, v_2 \in \tau$

$\Rightarrow$ every pair of nodes v_1, v_2 have an edge in $G_X(\tau)$ $\Rightarrow G_x(\tau)$ is a complete graph.

3.11 Result: $G_X(\tau)$ is not a complete graph if and only if the eccentricity of every node in $G_X(\tau)$ is never equal to red $G_X(\tau)$

Proof: Let $G_X(\tau)$ is not a complete graph

$\Rightarrow \exists$ atleast two nodes u, v such that $u \cup v \neq u \ or \ v$

$\Rightarrow d_{G_X(\tau)}(u,v) = 2 \ \left(\because d_{G_X(\tau)}(u,v) \leq 2\right)$

$\Rightarrow \underset{u \in V(G_X(\tau))}{Max} d_{G_X(\tau)}(u,v) = 2 \ \left(\because d_{G_X(\tau)}(u,v) \leq 2\right)$

$\Rightarrow ecc(u) = 2 \ \Rightarrow ecc(u) \neq rad\left(G_X(\tau)\right) \left(\because rad\left(G_X(\tau)\right) = 1\right)$

Conversely,

Let $u \in V\left(G_X(\tau)\right)$ such that $ecc(u) \neq rad\, G_x(\tau)$

$\Rightarrow ecc(u) \neq 1$

$\Rightarrow ecc(u) = 2 \ \left(\because ecc(u) = 1 \ or \ 2, \forall u \in V\left(G_X(\tau)\right)\right)$

$\Rightarrow \underset{v \in V(G_X(\tau))}{Max} d_{G_X(\tau)}(u,v) = 2$

$\Rightarrow$ Thus there exists u in $V\left(G_X(\tau)\right)$ such that $d_{G_X(\tau)}(u,v) = 2$

$\Rightarrow u \cup v \neq u \ or\, v \Rightarrow < u, v >$ is not an edge in $G_X(\tau)$

$\Rightarrow G_X(\tau)$ is not a complete graph.

3.12 Result: The number of triangles in $G_X(\tau)$ almost $\binom{|\tau|}{3} - \left\{\binom{n}{2} + \binom{n}{3}\right\}$ Where n is the order of the set of nodes whose eccentricity is two.

Proof: Let $L = \left\{v \in V\left(G_X(\tau)\right) | ecc(v) = 2\right\}$

Case-1: suppose $G_x(\tau)$ is a complete graph.

$\Rightarrow ecc(u) = 1, \forall u \in \tau$

$\Rightarrow v \notin L, \forall v \in \tau \Rightarrow L = \phi$

$\Rightarrow |L| = 0 \ \Rightarrow n = 0 \Rightarrow \binom{n}{2} = \binom{n}{3} = 0.$

Therefore, the number of triangles in $G_X(\tau)$ is at most

$$\binom{|\tau|}{3} - \left\{\binom{n}{2} + \binom{n}{3}\right\}.$$

Case-2: suppose $G_X(\tau)$ is not a complete graph. $\Rightarrow$ there exists atleast two nodes u, v in τ such that $ecc(u) = 2$ & $ecc(v) = 2 \Rightarrow u, v \in L \Rightarrow L \neq \phi$

Let $|\tau| = n$.

Sub Case-1: There exists at least two nodes in L have no edge $\Rightarrow$ these two nodes are not a vertices of same triangle $\Rightarrow$ thus the max number of pair in L are not a vertices of any triangle in $\binom{n}{2}$

Sub Case-2: There exists atleast three nodes in L have no edge among them. $\Rightarrow$ these three nodes are not vertices of same triangle. $\Rightarrow$ thus the maximum number of triangles of triplet in L are not a vertices of same triangle is $\binom{n}{3}$.

Thus, the number of triangles in $G_X(\tau)$ is atmost $\binom{|\tau|}{3} - \left\{\binom{n}{2} + \binom{n}{3}\right\}$

REFERENCES

1. Ackerman, Eyal (2009), "On the maximum number of edges in topological graphs with no four pairwise crossing edges", *Discrete & Computational Geometry* 41 (3): 365–375.
2. Douglas B. West , Introduction to Graph Theory, Pearson Education, Inc.,2006.
3. F. Harary, Graph theory, (Addison-Wesley, Reading, Mass., 1969).
4. George F. Simmons, Introduction to Topology and Modern Analysis, McGraw-Hill Book Company, Inc., 2004.
5. J. L. Kelley, General topology, Van Nostrand, Princeton, N. J., 1955.
6. Pach, János; Tóth, Géza (2003), "Disjoint edges in topological graphs", *IJCCGGT*: 133–140.
7. S. A. Naimpally, Essential fixed points and almost continuous functions, Ph. D. Thesis, Michigan State University, East Lansing, 1964.
8. W.T. Tutte, Connectivity in graphs, Math. Expotitions 15 {Univ. of Toronto Press, Toronto, Ont., 1967).
9. url: Bogdan Giuşcă. http://en.wikipedia.org/wiki/File:Konigsberg_bridges.png.
10. V. I. Ponomarev, A new space of closed sets and many-valued functions, Dokl. Akad. Nauk SSSR. 118(1958), 1081-1084.

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