

Minimal Weakly Closed Sets and Maximal Weakly Open Sets in Topological Spaces

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ABSTRACT

In this paper, a new class of sets called maximal weakly open sets and minimal weakly closed sets in topological Spaces are introduced which are subclasses of weakly open sets and weakly closed sets respectively. We prove that the complement of maximal weakly open set is a minimal weakly closed set and some properties of the new concepts have been studied.

Keywords: Minimal closed set, Maximal open set, Minimal weakly closed set, Maximal weakly open set.

Mathematics subject classification (2000): 54A05.

1. INTRODUCTION

In the year 2001 and 2003, F.Nakaoka and N.oda [1] [2] [3] introduced and studied minimal open (resp.minimal closed) sets which are sub classes of open (resp.closed) sets. The complements of minimal open sets and maximal open sets are called maximal closed sets and minimal closed sets respectively. In the year 2000 M. Sheik john [4] introduced and studied weakly closed sets and weakly open sets in topological spaces.

Definition: 1.1 [1] A proper non-empty open subset U of a topological space X is said to be minimal open set if any open set which is contained in U is \emptyset or U .

Definition: 1.2 [2] A proper non-empty open subset U of a topological space X is said to be maximal open set if any open set which is contained in U is X or U .

Definition: 1.3 [3] A proper non-empty closed subset F of a topological space X is said to be minimal closed set if any closed set which is contained in F is \emptyset or F .

Definition: 1.4 [3] A proper non-empty closed subset F of a topological space X is said to be maximal closed set if any closed set which is contained in F is X or F .

Definition: 1.5 [4] A subset A of (X, τ) is called weakly closed set if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X .

Definition: 1.6 [4] A subset A in (X, τ) is called weakly open set in X if A° is weakly closed set in X .

2. MAXIMAL WEAKLY OPEN SETS

Definition 2.1: A proper non-empty weakly open subset U of X is said to be maximal weakly open set if any weakly open set which is contained in U is X or U .

Remark 2.2: Maximal open sets and Maximal weakly open sets are independent of each other as seen from the following example.

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Example 2.3: Let $X = \{a, b, c\}$ be with the topology $\tau = \{X, \varnothing, \{a\}\}$

Open sets are $= \{X, \varnothing, \{a\}\}$

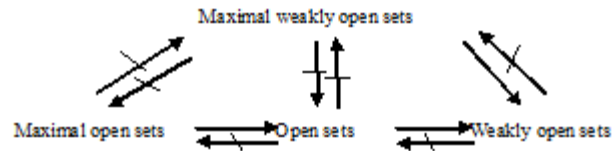
Maximal open sets are $= \{a\}$

Weakly open sets are $= \{X, \varnothing, \{a\}, \{a, b\}, \{a, c\}\}$

Maximal weakly open sets are $= \{\{a, b\}, \{a, c\}\}$

Here the set $\{a\}$ is a Maximal open set but not a Maximal weakly open set and the sets $\{a, b\}$ and $\{a, c\}$ are Maximal weakly open sets but not Maximal open sets.

Remark 2.4: From the Known results and by the above example we have the following implication.



Theorem 2.5:

- (i) Let U be a maximal weakly open set and W be a weakly open set then $U \cap W = \varnothing$ or $U \subset W$.
- (ii) Let U and V be maximal weakly open sets then $U \cap V = \varnothing$ or $U = V$

Proof:

(i): Let U be a maximal weakly open set and W be a weakly open set. If $U \cap W = \varnothing$, then there is nothing to prove but if $U \cap W \neq \varnothing$ then we have to prove that $U \subset W$. Suppose $U \cap W \neq \varnothing$ then $U \cap W \subset U$ and $U \cap W$ is Weakly open as the finite intersection of weakly open sets is a weakly open set. Since U is a maximal weakly open set, we have $U \cap W = U$ therefore $U \subset W$.

(ii): Let U and V be maximal weakly open sets suppose $U \cap V \neq \varnothing$ then we see that $U \subset V$ and $V \subset U$ by (i) therefore $U = V$.

Theorem 2.6: Let U be a maximal weakly open set if x is an element of U then $U \subset W$ for any open neighbourhood W of x .

Proof: Let U be a maximal weakly open set and x be an element of U . Suppose there exists an open neighbourhood W of x such that $U \not\subset W$ then $U \cap W$ is a weakly open set such that $U \cap W \subset U$ and $U \cap W \neq \varnothing$. Since U is a maximal weakly open set, We have $U \cap W = U$ that is $U \subset W$. This contradicts our assumption that $U \not\subset W$. Therefore $U \subset W$ for any open neighbourhood W of x .

Theorem 2.7: Let U be a maximal weakly open set, if x is an element of U then $U \subset W$ for any weakly open set W containing x .

Proof: Let U be a maximal weakly open set containing an element x . Suppose there exists a weakly open set W containing x such that $U \not\subset W$ then $U \cap W$ is a weakly open set such that $U \cap W \subset U$ and $U \cap W \neq \varnothing$. Since U is a maximal weakly open set, we have $U \cap W = U$ that is $U \subset W$. This contradicts our assumption that $U \not\subset W$. Therefore $U \subset W$ for any weakly open set W containing x .

Theorem 2.8: Let U be a maximal weakly open set then $U = \cap \{W : W \text{ is any weakly open set containing } x\}$ for any element x of U

Proof: By theorem 2.7 and from the fact that U is a weakly open set Containing x , We have $U \subset \cap \{W : W \text{ is any weakly open set containing } x\} \subset W$. Therefore we have the following result.

Theorem 2.9: Let U be a non-empty weakly open set then the following three conditions are equivalent.

- (i) U is a maximal weakly open set
- (ii) $U \subset w\text{-cl}(S)$ for any non-empty subset S of U .
- (iii) $w\text{-cl}(U) = w\text{-cl}(S)$ for any non-empty subset S of U .

Proof: (i) \implies (ii) Let U be a maximal weakly open set and S be a non-empty subset of U . Let $x \in U$ by theorem 2.7 for any weakly open set W containing x , $S \subset U \subset W$ which implies $S \subset W$. Now $S = S \cap U \subset S \cap W$. Since S is non-empty therefore $S \cap W \neq \varnothing$. Since W is any weakly open set containig x by one of the theorem, we know that, for an $x \in X$, $x \in w\text{-cl}(A)$ iff $\forall \text{ } \varnothing \neq V \ni x, V \cap A \neq \varnothing$. for any every weakly open set V Containing x that is $x \in U$ implies $x \in \text{cl}(s)$ which implies $U \subset w\text{-cl}(s)$ for any non-empty subset S of U .

(ii) \Rightarrow (iii) Let S be a non-empty subset of U that is $S \subset U$ which implies $w-cl(S) \subset w-cl(U) \rightarrow$ (a) Again from (ii) $U \subset w-cl(S)$ for any non-empty subset S of U Which implies $w-cl(U) \subset w-cl(w-cl(S)) = w-cl(S)$ i.e., $w-cl(U) \subset w-cl(S) \rightarrow$ (b), from (a) and (b) $w-cl(U) = w-cl(S)$ for any non empty subset S of U .

(iii) \Rightarrow (i) from (3) we have $w-cl(U) = w-cl(S)$ for any non-empty subset S of U . Suppose U is not a maximal weakly open set then there exist a non-empty weakly open set V such that $V \subset U$ and $V \neq U$. Now there exists an element $a \in U$ such that $a \notin V$ which implies $a \in V^c$ that is $w-cl(\{a\}) \subset w-cl(V^c) = V^c$, as V^c is a weakly closed set in X . It follows that $w-cl(\{a\}) \neq w-cl(U)$. This is contradiction to fact that $w-cl(\{a\}) = w-cl(U)$ for any non empty subset $\{a\}$ of U therefore U is a maximal weakly open set.

Theorem 2.10: Let V be a non-empty finite weakly open set, then there exists at least one (finite) maximal weakly open set U such that $U \subset V$.

Proof: Let V be a non-empty finite weakly open set. If V is a maximal weakly open set, we may set $U=V$. If V is not a maximal weakly open set, then there exists a (finite) weakly open set V_1 such that $\emptyset \neq V_1 \subset V$. If V_1 is a maximal weakly open set, we may set $U=V_1$. If V_1 is not a maximal weakly open set then there exists a (finite) weakly open set V_2 such that $\emptyset \neq V_2 \subset V_1$. Continuing this process we have a sequence of weakly open sets $V_k \dots \subset V_3 \subset V_2 \subset V_1 \subset V$. Since V is a finite set, this process repeats only finitely then finally we get a maximal weakly open set $U=V_n$ for some positive integer n .

Corollary 2.11: Let X be a locally finite space and V be a non-empty weakly open set then there exists at least one (finite) maximal weakly open set such that $U \subset V$.

Proof: Let X be a locally finite space and V be a non empty weakly open set. Let $x \in V$ since X is a locally finite space we have a finite open set V_x such that $x \in V_x$ then $V \cap V_x$ is a finite weakly open set. By theorem 2.10 there exist at least one (finite) maximal weakly open set U such that $U \subset V \cap V_x$ that is $U \subset V \cap V_x \subset V$. Hence there exists at least one (finite) maximal weakly open set U such that $U \subset V$.

Corollary 2.12: Let V be a finite maximal open set then there exist at least one (finite) maximal weakly open set U such that $U \subset V$

Proof: Let V be a finite minimal open set then V is a non-empty finite weakly open set, by theorem 2.10 there exist at least one (finite) maximal weakly open set U such that $U \subset V$.

Theorem 2.13: Let U and U_λ be maximal weakly open sets for any element λ of Λ . If $U \subset \bigcup_{\lambda \in \Lambda} U_\lambda$ then there exists an element $\lambda \in \Lambda$ such that $U = U_\lambda$.

Proof: Let $U \subset \bigcup_{\lambda \in \Lambda} U_\lambda$ then $U \cap (\bigcup_{\lambda \in \Lambda} U_\lambda) = U$ that is $\bigcup_{\lambda \in \Lambda} (U \cap U_\lambda) = U$ also by Theorem 2.5 (ii) $U \cap U_\lambda = \emptyset$ for any $\lambda \in \Lambda$ it follows that there exist an element $\lambda \in \Lambda$ such that $U = U_\lambda$.

Theorem 2.14: Let U and U_λ be maximal weakly open sets for any element $\lambda \in \Lambda$. if $U = U_\lambda$ for any element λ of Λ If $U = U_\lambda$ for any element λ of Λ then $(\bigcup_{\lambda \in \Lambda} U_\lambda) \cap U = \emptyset$.

Proof: Suppose that $(\bigcup_{\lambda \in \Lambda} U_\lambda) \cap U \neq \emptyset$ that is $\bigcup_{\lambda \in \Lambda} (U_\lambda \cap U) \neq \emptyset$. Then there exists an element $\lambda \in \Lambda$ such that $U \cap U_\lambda \neq \emptyset$ by theorem 2.5 (ii) we have $U = U_\lambda$, which contradicts the fact that $U \neq U_\lambda$ for any $\lambda \in \Lambda$ then $(\bigcup_{\lambda \in \Lambda} U_\lambda) \cap U = \emptyset$.

Theorem 2.15: Let U_λ be a maximal weakly open set for any element $\lambda \in \Lambda$ and $U_\lambda \neq U_\mu$ for any element λ and μ of Λ with $\lambda \neq \mu$ assume that $|\Lambda| > 2$. Let μ be any element of Λ then $(\bigcup_{\lambda \in \Lambda - \{\mu\}} U_\lambda) \cap U_\mu = \emptyset$.

Proof: Put $U = U_\mu$ in theorem 2.14, then we have the result.

Corollary 2.16: Let U_λ be a maximal weakly open set for any element $\lambda \in \Lambda$ and $U_\lambda \neq U_\mu$ for any element λ and μ of Λ with $\lambda \neq \mu$. If η a proper non-empty subset of Λ then $(\cup_{\lambda \in \Lambda} U_\lambda) \cap (\cup_{\gamma \in \eta} U_\gamma) = \emptyset$.

Theorem 2.17: Let U_λ and U_γ be maximal weakly open sets for any element $\lambda \in \Lambda$ and $\gamma \in \eta$ such that $U_\lambda \neq U_\gamma$ for any element λ of Λ then $\cup_{\gamma \in \eta} U_\gamma \not\subset (\cup_{\lambda \in \Lambda} U_\lambda)$.

Proof: Suppose that an element γ^1 of η satisfies $U_\lambda = U_{\gamma^1}$ for any element λ of Λ . if $\cup_{\gamma \in \eta} U_\gamma \subset (\cup_{\lambda \in \Lambda} U_\lambda)$, then we see

$U_{\gamma^1} \subset (\cup_{\lambda \in \Lambda} U_\lambda)$ by theorem 2.13 there exists an element λ of Λ such that $U_{\gamma^1} = U_\lambda$ which is a contradiction it

follows that $\cup_{\gamma \in \eta} U_\gamma \not\subset (\cup_{\lambda \in \Lambda} U_\lambda)$.

3. MINIMAL WEAKLY CLOSED SETS

Definition 3.1: A proper non-empty weakly closed subset F of X is said to be minimal weakly closed set if any weakly closed set which is contained in F is \emptyset or F

Remark 3.2: Minimal closed sets and Minimal weakly closed sets are independent each other as seen from the following implication.

Example 3.3: Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$ be a topological space.

Closed sets are $= \{X, \emptyset, \{b, c\}\}$

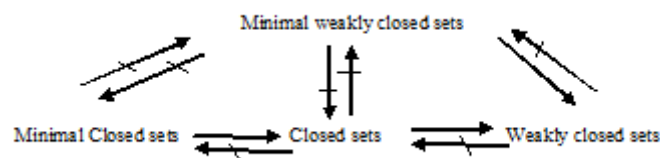
Minimal closed sets are $= \{b, c\}$

Weakly closed sets are $= \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$

Minimal weakly closed sets are $= \{\{b\}, \{c\}\}$

Here the set $\{b, c\}$ is a Minimal closed set but not a Minimal weakly closed set and the sets $\{b\}$ and $\{c\}$ are Minimal weakly closed sets but not Minimal closed sets.

Remark 3.4: From the known results and by the above example 3.3 we have the following implication.



Theorem 3.5: A proper non-empty subset F of X is minimal weakly closed set iff $X-F$ is a maximal weakly open set.

Proof: Let F be a minimal weakly closed set, suppose $X-F$ is not a maximal weakly open set then there exists a weakly open set $U \neq X-F$ such that $\emptyset \neq U \subset X-F$ that is $F \subset X-U$ and $X-U$ is a weakly closed set. This contradicts our assumption that F is a maximal weakly open set. Conversely, let $X-F$ be a maximal weakly open set. Suppose F is not a minimal weakly closed set then there exist a weakly closed set $E \neq F$ such that $F \subset E \neq X$ that is $\emptyset \neq X-E \subset X-F$ and $X-E$ is a weakly open set. This Contradicts our assumption that $X-F$ is a maximal weakly open set. Therefore F is a minimal weakly closed set.

Theorem 3.6:

(i): Let F be a minimal weakly closed set and W be a weakly closed set Then $F \cup W = X$ or $W \subset F$.

(ii): Let F and S be minimal weakly closed sets then $F \cup S = X$ or $F = S$

Proof:

(i): Let F be a minimal weakly closed set and W be a Weakly closed set if $F \cup W = X$ then there is nothing to prove but if $F \cup W \neq X$ then we have to prove that $W \subset F$. Suppose $F \cup W \neq X$ then $F \subset F \cup W$ and $F \subset W$ is weakly closed as the finite union of weakly closed set is a weakly closed set we have $F \cup W = X$ Therefore $F \cup W = F$ which implies $W \subset F$.

(ii): Let F and S be minimal weakly closed sets. Suppose $F \cup S \neq X$ then we see that $F \subset S$ and $S \subset F$ by (i) therefore $F = S$.

Theorem 3.7: Let F be a minimal weakly closed set. If x is an element of F then for any weakly closed set S containing x , $F \cup S = X$.

Proof: Proof is similar to 2.7 theorem.

Theorem 3.8: Let $F_\alpha, F_\eta, F_\gamma$ be minimal weakly closed sets such that $F_\alpha \neq F_\eta$ if $F_\alpha \cap F_\eta \subset F_\gamma$. then either $F_\alpha = F_\gamma$ or $F_\eta = F_\gamma$.

Proof: Given that $F_\alpha \cap F_\eta \subset F_\gamma$, if $F_\alpha = F_\gamma$ then there is nothing to prove but if $F_\alpha \neq F_\gamma$ then We have to prove $F_\eta = F_\gamma$.

$$\begin{aligned} \text{Now we have } F_\eta \cap F_\gamma &= F_\eta \cap (F_\gamma \cap X) \\ &= F_\eta \cap (F_\gamma \cap (F_\alpha \cap F_\eta)) \text{ (by theorem 3.6 (ii))} \\ &= F_\eta \cap (F_\gamma \cap F_\alpha) \cup (F_\gamma \cap F_\eta) \\ &= (F_\eta \cap F_\gamma \cap F_\alpha) \cup (F_\eta \cap F_\gamma \cap F_\eta) \\ &= (F_\alpha \cap F_\eta) \cup (F_\gamma \cap F_\eta) \text{ (by } F_\eta \cap F_\gamma \cap F_\alpha) \\ &= (F_\alpha \cup F_\gamma) \cap F_\eta \\ &= X \cap F_\eta \text{ (since } F_\alpha \text{ and } F_\gamma \text{ are minimal weakly closed sets by thm 3.6(ii) } F_\alpha \cup F_\gamma = X \\ &= F_\eta \end{aligned}$$

that is $F_\eta \cap F_\gamma = F_\eta$ implies $F_\eta \subset F_\gamma$, since F_η, F_γ are minimal weakly closed sets we have $F_\eta = F_\gamma$.

Theorem 3.9: Let $F_\alpha, F_\eta, F_\gamma$ be minimal weakly closed sets which are different from each other then $(F_\alpha \cap F_\eta) \not\subset (F_\alpha \cap F_\gamma)$.

Proof: Let $(F_\alpha \cap F_\eta) \subset (F_\alpha \cap F_\gamma)$ which implies $(F_\alpha \cap F_\eta) \cup (F_\gamma \cap F_\eta) \subset (F_\alpha \cap F_\gamma) \cup (F_\gamma \cap F_\eta)$ which implies $(F_\alpha \cup F_\gamma) \cap (F_\gamma \cup F_\eta) \cap (F_\alpha \cup F_\eta)$ since by theorem 3.6 (ii) $F_\alpha \cap F_\gamma = X$ and $F_\alpha \cap F_\eta = X$ which implies $X \cap F_\eta \subset F_\gamma \cap X$ which implies $F_\eta \subset F_\gamma$. From the definition of minimal weakly closed set it follows that $F_\eta = F_\gamma$. This is contradiction to the fact that Let $F_\alpha, F_\eta, F_\gamma$ are different from each other. Therefore $(F_\alpha \cap F_\eta) \not\subset (F_\alpha \cap F_\gamma)$.

Theorem 3.10: Let F be a minimal weakly closed set and x be an element of F then $F = \bigcup \{S : S \text{ is a weakly closed set containing } x \text{ such that } F \cup S \neq X\}$

Proof: By theorem 3.7 and from the fact that F is a weakly closed set containing x we have $F \subset \bigcup \{S : S \text{ is a weakly closed set containing } x \text{ such that } F \cup S \neq X\} \subset F$ therefore we have the result.

Theorem 3.11: Let F be a Proper non-empty co-finite weakly closed subset then there exists (co-finite) minimal weakly closed set E such that $F \subset E$.

Proof: Let F be a non-empty co-finite weakly closed set. If F is a minimal weakly closed set, we may set $E = F$. If F is not a minimal weakly closed set, then there exists a (co-finite) weakly closed set F_1 such that $F \subset F_1 \neq X$. If F_1 is a minimal weakly closed set, we may set $E = F_1$. if F_1 is not a minimal weakly closed set, then there exists a (co-finite) weakly closed set F_2 such that $F \subset F_1 \subset F_2 \neq X$ continuing this process we have a sequence of weakly closed sets $F \subset F_1 \subset F_2 \subset \dots \subset F_k \subset \dots$ since F is a co-finite set, this process repeats only finitely then finally we get a maximal weakly open set $E = E_n$ for some positive integer n .

Theorem 3.12: Let F be a minimal weakly closed set. If x is an element of $X - F$ then $X - F \subset E$ for any weakly closed set containing set E containing x

Proof: Let F be a minimal weakly closed set and $x \in X - F$. $E \not\subset F$ for any weakly closed set E containing x then $E \cup F = X$ by theorem 3.6(ii). Therefore $X - F \subset E$.

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