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# SOME SPECIAL NEAR – RINGS

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**ABSTRACT**

*In [4] ([5]) we defined a right near – ring N to be $\beta_1(\beta_2)$ if xNy = Nxy(xNy = xyN) for all x, y in N. Following these we make an attempt in this paper to study the properties of those near – rings which satisfy the conditions xNy = yxN and xNy = Nyx.*

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## 1. INTRODUCTION

A right near – ring is an algebraic system (N,+,·) with two binary operations '+'and '·' such that
(i) (N, +) is a group with 0 as its identity.
(ii) (N, ·) is a semigroup and
(iii) (x + y) z = xz + yz for all x, y, z in N.

Throughout this paper, N stands for a right near – ring (N, +, ·) with at least two elements . Obviously, 0n = 0 for all n in N. As in [2], a subgroup (M, +) of (N, +) is called (i) a left N – subgroup of N if MN⊂M, (ii) an N – subgroup of N if NM⊂M and (iii) an invariant N – subgroup of N if M satisfies both (i) and (ii). Again in [2], N is defined to be weak commutative if xyz= xzy for all x, y, z in N. The concept of mate function in N has been introduced in [6] with a view to handling the regularity structure with considerable ease. A map '*f* 'from N into N is called (i) a mate function for N if *x = xf(x) x*. (ii) a $P_3$ mate function if in addition, *xf(x) = f(x)x* for all x in N. By identity 1of N, we mean only the multiplicative identity of N.

Basic concepts and terms used but left undefined in this paper can be found in [2].

## 2. NOTATIONS

(i) E denotes the set of all idempotents of N. (e in N is called an idempotent if $e^2 = e$)
(ii) L denotes the set of all nilpotents of N. (a in N is nilpotent if $a^k = 0$ for some positive integer k)
(iii) $N_d$ = {n∈N/ n(x+y) = nx + ny for all x, y in N} - set of all distributive elements of N.
(iv) C(N) = {n∈N/ nx = xnfor all x in N}- centre of N.
(v) $N_0$ = {n∈ N/ n0 = 0} – zero -symmetric part of N.

## 3. PRELIMINARY RESULTS

We freely make use of the following results and designate them as R(1), R(2),...etc

**R(1)**N has no non – zero nilpotent elements if and only if $x^2 = 0 \Rightarrow x = 0$ for all x in N. (Problem 14, p.9 of [3]).

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**R(2)** If $f$ is a mate function for N, then for every x in N, $xf(x)$, $f(x)x \in E$ and $Nx = Nf(x)x$, $xN = xf(x)N$. (Lemma 3.2 of [6])

**R(3)** If L={0} and N=$N_0$ then (i) xy = 0 ⇒yx = 0 for all x, y in N (ii) N has Insertion of Factors Property – IFP for short (i. e) for x, y in N, xy=0 ⇒ xny=0 for all n in N. If Nsatisfies (i) and (ii) then N is said to have (∗, IFP) (Lemma 2.3 of [6])

**R(4)** Any weak commutative near – ring with a left identity is pseudo commutative (i.e) xyz = zyx for all x, y, z in N. (Proposition 2.8 of [7])

**R(5)** N has strong IFP if and only if for all ideals I of N, and for all x, y, n∈N, xy∈ I⇒ xny ∈ I (Proposition 9.2,p. 289 of [2]).

## 4. $\beta_3$ AND $\beta_4$ NEAR – RINGS

In this section we define $\beta_3$ and $\beta_4$ near – rings and give certain examples of these new concepts.

**Definition: 4.1** Let N be a right near– ring. If for all x, y in N, xNy =yxN (xNy = Nyx) then we say N is a $\beta_3$ near – ring ($\beta_4$ near – ring).

**Examples: 4.2**
(a) Let (N, +) be the Klein's four group with multiplication defined as per scheme 7, p.408 of Pilz [2]

| · | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | a |
| b | 0 | 0 | b | b |
| c | 0 | a | b | c |

This near – ring N is $\beta_3$ as well as $\beta_4$. Here, the identity function serves as a mate function.

(b) Then near – ring (N, +, ·) where (N,+) is defined on Klein's four group with N={0,a,b,c} and '·' defined as per scheme 14, p.408 of Pilz [2]

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | c |
| b | 0 | 0 | 0 | 0 |
| c | 0 | a | 0 | c |

is neither $\beta_3$ (since aNc≠ caN) nor $\beta_4$ (since aNc≠ Nca ). It is worth noting that this near – ring does not admit mate functions.

(c) Let N be an arbitrary near –ring. Let I be the ideal generated by {anb – ban′ / a, b, n, n′are in N}. The factor near – ring $\overline{N}$ = N/I is a $\beta_3$ near – ring.

(d) Let N be an arbitrary near – ring. Let I be the ideal generated by {an b – n′ba/a, b, n, n′ are in N}.The factor near – ring $\overline{N}$=N/I is a $\beta_4$ near – ring

## 5. $\beta_3$ near – Ring

In this section we study some of the important properties of a $\beta_3$ near – rings and give a complete characterization of such near – rings.

**Proposition: 5.1** If N is a $\beta_3$near – ring, then xNx = $x^2$N for all x in N.

**Proof:** When N is a$\beta_3$near – ring, by definition, for all x, y in N, xNy = yxN (1)

The result follows by replacing y by x in (1)

**Remark: 5.2** The converse of Proposition 5.1 is not true. For example, we consider the near- ring (N, +, ·) where (N, +) is the Klein's four group {0, a, b, c} and '·' is defined as per scheme 3,p.408 of Pilz [2]

| . | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | 0 | a | a |
| b | 0 | 0 | b | b |
| c | 0 | 0 | c | c |

satisfies the condition $xNx= x^2N$ for all x in N. But it is not a $\beta_3$ near – ring. [Since $aNb \neq baN$].

**Proposition: 5.3** Let N be a $\beta_3$ near – ring with identity 1. Then we have the following:
(i) N is zero symmetric.
(ii) N has strong IFP.
(iii) Every N –subgroup of N is invariant.
(iv) Every left N–subgroup of N is an N - subgroup if $N = N_d$.

**Proof:** Let N be a $\beta_3$near – ring. Then for all x, y in N, xNy = yxN (1)

(i) Putting x=1 in (1), we get, 1Ny =y1N for all y in N. When y=0, N0 =0N= {0}. It follows that N is zero-symmetric.

(ii) Let I be an ideal of N (2)

and let xy∈ I.Now, y1x∈yNx [since 1∈N] = xyN [by (1)] ∈ IN⊆I[by (2)].

Therefore, yx∈I (3)

Now, for any n in N, we have xny∈xNy = yxN [by (1)]∈IN ⊆ I [by(2)]. From R(5), it follows that N has strong IFP.

(iii) Let S be any N-subgroup of N. Then S =$\sum_{x\in s}$ Nx (4)

Now, NxN = Nx1N = N1Nx [by (1)] = NNx⊂Nx⇒ NxN ⊂ Nx (5)

Therefore, SN = [$\sum_{x\in s}$ Nx].N [by (4)] ⊂ $\sum_{x\in s}$ NxN ⊂ $\sum_{x\in s}$ Nx [by(5)] = S[by(4)].

Consequently S is an invariant N-subgroup.

(iv) Let S be any left N-subgroup of N where N = $N_d$. Then S = $\sum_{x\in s}$ xN (6)

Now, NxN = 1NxN = x1NN [by (1)] = xNN⊂ xN [since SN⊂ S] ⇒ NxN ⊂ xN (7)

Hence NS= N[$\sum_{x\in s}$ xN][by(6)]⊂ $\sum_{x\in s}$ NxN ⊂ $\sum_{x\in s}$ xN[by(7)] = S[by(6)]. Consequently, every left N – subgroup is an N – subgroup.

**Proposition: 5.4** Any homomorphic image of a $\beta_3$ near–ring is also a $\beta_3$ near–ring.

**Proof:** Straight forward.

**Theorem: 5.5** Every $\beta_3$ near – ring N is isomorphic to a subdirect product of subdirectly irreducible $\beta_3$ near – rings.

**Proof:** By Theorem 1.62, p. 26 of Pilz [2], N is isomorphic to a subdirect product of subdirectly irreducible near–rings $N_i$'s and each $N_i$ is a homomorphic image of N under the projection map $\pi_i$. The rest of the proof is taken care of by Proposition 5.4.

We furnish below a necessary and sufficient condition for a $\beta_3$ near – ring to admit mate functions.

**Proposition: 5.6** Let N be the $\beta_3$ near - ring. Then N admits mate functions if and only if x∈ $x^2$N for all x in N.

**Proof:** We first observe from Proposition 5.2 that, since N is $\beta_3$, xNx=$x^2$N for all x in N (1)

For the 'only if part', we assume that *f* is a the mate function for N. Then for all x in N, $x=xf(x)\,x\in xNx$. It follows that $x \in x^2N$.

For the 'if part 'let x∈$x^2$N for all x in N. Appealing to (1) we get x=xnx for some n in N. By setting n=*f*(x), we see that *f* is a mate function for N.

 

In the following results we assume that N has a mate function.

**Theorem: 5.7** Let $N=N_d$ be a zero – symmetric $\beta_3$ near – ring with a mate function. Then we have,

(i) L={0}
(ii) N has (*, IFP)
(iii) $E\subset C(N)$
(iv) $xN\cap yN=yNxN = xyN$ for all x, y in N.

**Proof:** (i) Since *f* is a mate function for N, Proposition 5.6 demands that $x\in x^2N$ for all x in N. Therefore, $x=x^2n$ for some n in N. Suppose $x^2 = 0$. Clearly then x = 0. Now, R(1) guarantees that L={0}.

(ii) By (i) L= {0}. Now, R(3) guarantees that N has (∗,IFP).

(iii) Let $e \in E$. Since N is $\beta_3$, eNe = eeN = eN. Therefore, for any n in N, ene = eu and en = eve for some u, v in N.

Now, ene = (eu)e and (en)e = eve. Thus ene = en for all n in N (1)

We also have, e(ne-ene) = 0 [since $N=N_d$] $\Rightarrow$ ene(ne-ene) = 0[by(ii)].And ne(ne-ene) = n.0 = 0 [Since $N = N_o$].

Consequently, $(ne-ene)^2 = 0$ and (i) guarantees ne-ene = 0. Therefore, ene = ne for all n in N (2)

Combining (1) and (2) we get en = ne for all n in N. Thus $E\subset C(N)$.

(iv) First we show that for any left N-subgroups A and B of N, $A\cap B = BA$. By Proposition 5.3 (iv), A and B are N-subgroups of N. Now, for $x\in A$ and $y\in B$, $yx\in BN\subset B$.Therefore, $BA\subset B$ (3)

Also, $yx\in NA\subset A$. Hence $BA\subset A$ (4)

Combining (3) and (4), $BA\subset A\cap B$ (5)

On the other hand, if $z\in A\cap B$ then since '*f*' is a mate function for N, $z= zf(z)z\in(BN)A \subset BA$.

Consequently, $A\cap B \subset BA$ (6)

Combining (5) and (6) $A\cap B = BA$ for all left N–subgroups A, B of N. We know that, xN and yN are left N- subgroups of N.

Therefore, $xN\cap yN=yNxN$ for all x, y in N. (7)

On the other hand, if $y \in N$, then, Since *f* is a mate function for N, $yN = (yf(y)y)N \subset yNN \Rightarrow xyN\subset xyNN = yNxN$ [since N is $\beta_3$].

Therefore, $xyN\subset yNxN$ (8)

For the reverse inclusion, yNxN = xyNN [since N is $\beta_3$]$\subset$ xyN. Therefore, $yNxN\subset xyN$ (9)

From (8) and (9), yNxN = xyN (10)

for all x, y in N. Combining (7) and (10) we get $xN\cap yN = yNxN = xyN$ for all x, y in N.

We furnish below a characterization theorem for $\beta_3$near –ring.

**Theorem: 5.8** Let $N=N_d$ be a zero–symmetric near-ring with a mate function *f*. Then N is$\beta_3$ if and only if *xyN = yxN* for all x, y in N and $E\subset C(N)$.

**Proof:** For the 'only if part', first we observe that $E\subset C(N)$ (1)

[by Theorem 5.7 (iii)]. Since *f* is a mate function for N. Now, $xyN = xN\cap y N$ [by Theorem 5.7(iv)] $= yN\cap xN = yxN$ [by Theorem 5.7(iv)].

For the 'if part', first we show that' $f$ 'is a $P_3$ mate function for any $x\in N$ we have $x=xf(x)x=x^2f(x)$ [Since $E\subset C(N)$]$\Rightarrow x(f(x)x - xf(x))=0 \Rightarrow xf(x)(f(x)x - xf(x))=0$ [by Theorem 5.7(ii)] and $f(x)x(f(x)x - xf(x))=f(x).0=0$ [since $N=N_0$]. Consequently, $(f(x)x - xf(x))^2 = 0$ and hence $xf(x) = f(x)x$ ………..…..(7) for all x in N [by R(1)].

Hence $f$ is a $P_3$matefunction.Now, $xNy = xNf(y)y = xNyf(y)$[by (7)]$=x[yf(y)N]$[Since $E\subset C(N)$] $= xyN = yxN$ [by hypothesis].

## 6. $\beta_4$ near – ring

Throughout this section N denotes a $\beta_4$ near–ring. In this section we study some of the important properties.

**Proposition: 6.1** If N is a $\beta_4$ near – ring, then $xNx = Nx^2$ for all x in N.

**Proof:** When N is a $\beta_4$ near-ring, by definition for all x, y in N, xNy = Nyx (1)

The result follows by replacing y by x in (1).

**Remark: 6.2** The converse of Proposition 6.1 is not true. For example, Consider the near – ring (N, +, ·) where (N, +) is the Klein's four group {0, a, b, c} and '·' is defined as per scheme 13, p.408 of Pilz [2]

| · | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | b | c |
| b | 0 | 0 | 0 | 0 |
| c | 0 | a | b | c |

Satisfies the condition $xNx=Nx^2$for all x in N. But it is not a $\beta_4$ near – ring [since aNb≠Nba].

**Proposition: 6.3** If N is a $\beta_4$ near – ring, then NxNy = NyNx for all x, y in N.

**Proof:** Since N is a $\beta_4$ near – ring, we have xNy = Nyx (1)

for all x, y in N. Now, for any x, y, n in N, (nx) Ny⊂ Nx Ny ⇒ Ny(nx) ⊂ NxNy[by(1)]⇒ NyNx ⊂NxNy (2)

On the other hand, (ny) Nx⊂ NyNx⇒Nx(ny) ⊂ NyNx [by(1)] ⇒NxNy⊂ NyNx (3)

Combining (2) and (3), NxNy=NyNx for all x, y in N.

**Remark: 6.4** The converse of Proposition 6.4 is not valid. For example, the near– ring (N, +, ·) where (N, +) is the Klein's four group {0, a, b, c} and '·' is defined as per scheme 20, p.408 of Pilz [2]

| · | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | a | a | a | a |
| b | 0 | a | b | c |
| c | a | 0 | c | b |

satisfies the condition NxNy = NyNx for all x, y in N. But it is not a $\beta_4$ near – ring [since 0Na≠Na0]

**Theorem: 6.5** Every weak commutative near – ring with identity is $\beta_4$.

**Proof:** Let N be a weak commutative near – ring (1)

Let x, y∈N. For any n∈N, xny = xyn[by(1)] = nyx [byR(4)] ∈Nyx. Thus xNy⊂Nyx
(2)

On the other hand, for $n_1$ in N, $n_1yx = xyn_1$ [by R(4)] $= xn_1y$[by(1)] ∈ xNy.Consequently, Nyx⊂ xNy (3)

Combining (2) and (3) we get, N is a $\beta_4$ near – ring.

**Proposition: 6.6** In a $\beta_4$ near – ring, if $N=N_d$ then xM=Mx for all N - subgroups M of N.

**Proof:** Let N be a $\beta_4$ near – ring. Then for any x, y in N, xNy = Nyx (1)

Let $M=\sum_{y\in M} Ny$ be an N - subgroup of N. (2)

Then $xM = x\sum_{y\in M} Ny = \sum_{y\in M} x(Ny) = \sum_{y\in M} Nyx$ [by(1)] $= [\sum_{y\in M} Ny]x = Mx$ [by(2)].

In view of Theorem 5.7 it is worth mentioning that a $\beta_4$ near – ring also possesses certain properties which are satisfied by a $\beta_3$ near – ring as in the following result.

**Theorem: 6.7** Let N be a zero-symmetric $\beta_4$ near – ring with mate function. Then we have,
(i) L={0}
(ii) N has (∗, IFP)
(iii) E⊂C(N)
(iv) Nx∩Ny = NyNx= Nxy for all x, y in N.

**Proof:**
(i) Since $f$ is a mate function for N, we have $x=xf(x)\ x\in xNx$. It follows that $x\in Nx^2$ [by Proposition 6.1]. Therefore, $x=nx^2$ for some n in N. Suppose $x^2 = 0$. Clearly, then $x=0$. Now, R(1) guarantees that L={0}.

(ii) By (i) L={0}. Now, R(3) guarantees that N has (∗,IFP).

(iii) Let e∈E. Since N is$\beta_4$, eNe = Nee = Ne. Therefore for any n in N, ene = ue and ne = eve for some u, v in N.

Now, ene = eue and ene = eve. Thus ene = ne for all n in N (1)

We also have, (en – ene)e = 0⇒e(en – ene)=0 [by(ii)]⇒ene(en – ene)= 0[by(ii)].

Consequently, $(en – ene)^2 = 0$ and (i) guarantees ene=en for all n in N (2)

Combining (1) and (2) we get ne=en for all n in N. Thus E⊂C(N).

(iv) The result follows by replacing the left N–subgroups xN, yN in the proof of Theorem 5.7(iv) by the right N–subgroups Nx, Ny respectively.

We furnish below a characterization theorem for $\beta_4$near – ring.

**Theorem: 6.8** Let N be a zero – symmetric near – ring with a mate function f. Then N is $\beta_4$ if and only if Nxy = Nyx for all x, y in N and E⊂C(N).

**Proof:** The proof is similar to that of Theorem 5.8.

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