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THE ELEMENT IDEAL GRAPHS

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ABSTRACT

Let R be a commutative ring with identity and let x be an element of R. The Element Ideal Graph $\Gamma_x(R)$ is a graph whose vertex set is the set of nontrivial ideals of R and two vertices I and J are adjacent if and only if $x \in IJ$. In this paper we introduce the concept of element ideal graph we give the main properties of such graph, we also investigate the interplay between the graph theoretic properties of $\Gamma_x(R)$ and the annihilating ideal graph AG(R).

Keywords: *Zero divisor graph, annihilating ideal graph and element ideal graph.*

2010 Mathematics subject classification: *05C25.*

INTRODUCTION

Let R be a commutative ring with identity, and let Z(R) be its set of zero divisors. We associate a simple graph Γ(R) to R with vertices $Z^*(R) = Z(R) \setminus \{(0)\}$, the set of all non-zero zero divisors of R, and for distinct $x, y \in Z^*(R)$, the vertices x and y are adjacent if and only if xy=0. Obviously Γ(R) is empty if R is an integral domain.

The zero divisor graph of a commutative ring was introduced in [4], and further studied in [1, 2, 3, 9, 10]. The annihilating ideal graph AG(R) is a graph with vertex set $AG^*(R) = AG(R) \setminus \{(0)\}$ such that there is an edge between vertices I and J if and only if $IJ = (0)$. The idea of annihilating ideal graph was introduced by Behboodi and Rakeei in [5, 6].

In the present paper we define and study a new kind of graph of ideal vertices. The element Ideal Graph $\Gamma_x(R)$ is a graph whose vertex set is the set of nontrivial ideals of R and two ideal vertices I and J are adjacent if and only if $x \in IJ$. Obviously $\Gamma_x(R)$ is empty graph if R is a field.

1. BACKGROUND

In this section we state some definitions and theorems that we need in our work.

Definition 1.1 [7]: The ideals I and J of the ring R are said to be comaximal if I+J=R.

Definition 1.2 [8]:

1. The distance d(u, v) between a pair of vertices u and v of the graph Γ is the minimum of the lengths of the u—v paths of Γ.
2. The degree of the vertex *a* in the graph Γ is the number of edges incident to *a*.
3. The graph Γ is called a plane graph if it can be drawn on a plane in such a way that any two of its edges either meet only at their end vertices or do not meet at all. A graph which is isomorphic to a plane graph is called a planar graph.
4. A bipartite graph is one whose vertex set is partitioned into two disjoint subsets in such a way that the two end vertices for each edge lie in distinct partition. The complete bipartite graph with exactly two partitions of order m and n is denoted by K_{mn}.
5. A complete subgraph Kn of a graph Γ is called a clique, and cl(Γ) is the clique number of Γ, which is the greatest integer $r \geq 1$ such that $K_r \subseteq \Gamma$.

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Theorem 1.3 [8, P. 96]: (Kuratowsky Theorem) A graph Γ is planar if and only if it does not contains a graph homomorphic with K_5 or K (3, 3).

Theorem 1.4 [5, P.4]: Let R be a ring. Then the following statements are equivalent.
(1) AG(R) is a finite graph.
(2) R has only finitely many ideals.
(3) Every vertex of AG(R) has finite degree.
Moreover, AG(R) has n (n ≥ 1) vertices if and only if R has only n nonzero proper ideals.

Theorem 1.5 [5, P.8]: For every ring R, the annihilating-ideal graph AG(R) is connected and diam (AG(R)) ≤ 3. Moreover, if AG(R) contains a cycle, then gr(AG(R)) ≤ 4.

2. THE ELEMENT IDEAL GRAPH

In this section we introduce the notion of element ideal graph, we give some of its basic properties and provide some examples.

Definition 2.1: Let R be a commutative ring with identity and let x∈R. The element ideal graph is a graph whose vertex set is nontrivial ideals of R, and two of its vertices I and J are adjacent if and only if $x \in$ I J. We denote the element ideal graph by Γ_x(R).

We shall write I**—**J to denote for I and J to be adjacent.

Before stating our results, the following example is needed.

Example 1: Let Z_{12} be the ring of integers modulo 12. The graph of $\Gamma_6(z_{12})$ consists of the only edge (2) **—** (3).

The following result is an easy consequence of definition 2.1.

Lemma 2.2: All vertices of Γ_x(R) contain x.

Example 2: Let Z be the ring of integers. Clearly the ideal vertices (2), (3), (6) and (9) of Γ_{18}(Z) contain 18.

The next result illustrates that the element x is a zero divisor under a condition for the vertex set of the graph $\Gamma_x(R)$.

Theorem 2.3: If Z(R) is an ideal vertex of $\Gamma_x(R)$, then x is a zero divisor of R.

Proof: Since Z(R) is an ideal vertex of $\Gamma_x(R)$, then there exists an ideal I of R such that I**—**Z(R) is an edge of $\Gamma_x(R)$.This means that $x \in I \cdot Z(R)$, and hence $x = \sum_i a_i\, b_i$ for some $a_i \in I$ and $b_i \in Z(R)$. Since x$\neq 0$, then $b_i \neq 0$ and there exists $c_i \in Z^*(R)$ such that $b_i c_i = 0$, it follows that x$c_i = 0$. This shows that x is a zero divisor of R.

Example 3: Let Z_{16} be the ring of integers modulo 16. Clearly 8 is a zero divisor of Z_{16}, $Z(Z_{16})$=(2) and 8 ∈(2)(4). Then $Z(Z_{16})$=(2) is a vertex of $\Gamma_8(Z_{16})$.

The converse of Theorem2.3 may not be true in general, as the following example shows.

Example 4: Let Z_{12} be the ring of integers modulo 12. Clearly 8 is a zero divisor of Z_{12}, while (Z_{12})={0,2,3,4,6,8,9,10} is not an ideal vertex of $\Gamma_8(Z_{12})$

Recall that a ring R is called reduced if R has no non-zero nilpotent element. The next result shows that all vertices of the graph $\bigcup_{x\neq 0}\Gamma_x$(R) are looped under a sufficient condition for R.

Proposition 2.4: If R is a reduced ring, then all ideal vertices of $\bigcup_{x\neq 0}\Gamma_x$(R) are looped.

Proof: Suppose that $\bigcup_{x\neq 0}\Gamma_x$(R) has no loop at an ideal vertex I. Then I I= (0). This implies that $a^2 = 0$ for every$a \in I$. Since R is a reduced ring, then a=0 for every $a \in I$. This contradicts the fact that I is a nontrivial ideal. Therefore all ideal vertices of $\bigcup_{x\neq 0}\Gamma_x$(R) are looped.

Example 5: Let Z be the ring of integers. Obviously Z is a reduced ring and $(a)\ (a) \neq (0)$ for every nontrivial ideal (a) of Z, therefore $\bigcup_{x\neq 0}\Gamma_x$(Z) has a loop at every its vertex (a).

If R is not reduced ring, then $\bigcup_{x\neq 0}\Gamma_x$(R) may have a non-looped vertex. The following example illustrates it.

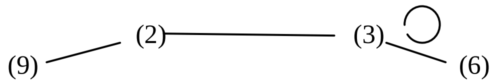
Example 6: Let Z_{12} be the ring of integers modulo 12. Clearly there is no x such that x∈ (6) (6). This means that (6) — (6) is not an edge in $\Gamma_x(Z_{12})$ for every x∈$Z_{12}\setminus\{0\}$. This yields that $\bigcup_{x\neq 0}\Gamma_x\ (Z_{12})$ has a non-looped vertex (6).

We next turn to give the following result.

Proposition 2.5: If I and J are adjacent ideal vertices in Γ_x(R), and K is an ideal containing J, then K is an ideal vertex of Γ_x(R) and I is adjacent to K in Γ_x(R).

Proof: The proof follows directly from the definition of Γ_x(R)**.**

Example 7: Let Z the ring of integers.



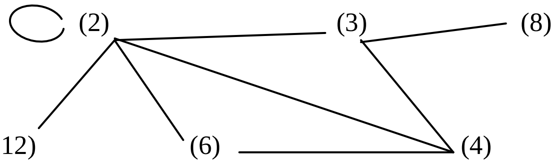
Clearly (2) is adjacent to (9) in Γ_{18}(Z), and (9) ⊆ (3). So (2) is also adjacent to (9) in Γ_{18}(Z).

The following Corollaries are immediate from Proposition 2.5.

Corollary 2.6: If I— J is an edge of $\Gamma_x(R)$ with deg(J) = n, then any chain of ideal vertices of $\Gamma_x(R)$ which starts with I, has length at most n.

Proof: Let $I=I_1\subseteq I_2 \subseteq\ldots\subseteq I_m$ be a chain of vertices of $\Gamma_x(R)$. Then by Proposition2.5, J is adjacent to vertices I_1 , I_2 ,…, I_m. This means that the ideal vertex J has degree at least m, but deg(J)=n, so m≤n. Therefore any chain of vertices of $\Gamma_x(R)$ which starts with I, has length at most n.

Example 8: Let Z be the ring of integers.

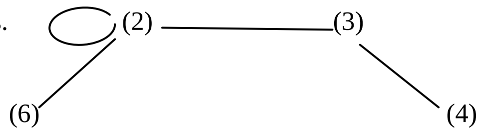


Clearly (2) is adjacent to (6) in Γ_{24}(Z), deg((2))=2 and the length of (6) ⊆ (3) is not greater than two.

Corollary 2.7: For two distinct ideal vertices I and J of Γ_x(R), if I⊆J then deg (I)≤deg(J).

Proof: The prove is trivial.

Example 9: Let Z be the ring of integers.



Γ_{12}(Z)

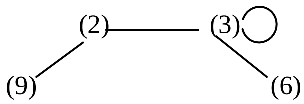
Clearly (6) ⊆(3) and deg ((6)) =1<2=deg ((3)).

The next result demonstrates that the vertex set of the element ideal graph is always contain a maximal ideal of R.

Theorem 2.8: If Γ_x(R) is non-empty, then its vertex set contains a maximal ideal of R, moreover if I is an ideal vertex of Γ_x(R) which is not maximal ideal, then |Γ_x(R) |>1.

Proof: Let I be an ideal vertex of Γ_x(R). Then there exists an ideal J such that I— J is an edge in Γ_x(R). If I is a maximal ideal, the theorem holds. Now suppose that I is not a maximal ideal of R .Then I contains in a maximal ideal of R say M. Then there exists an ideal J of R such that I— J is an edge of Γ_x(R). By Proposition 2.5, J is adjacent to M in Γ_x(R). This means that M is an ideal vertex of Γ_x(R). Clearly Γ_x(R) contains a path I— J— M (The vertex J may be equal to one of I and M). Thus |Γ_x(R) |>1.

Example 10: Consider the graph Γ_{18}(Z).



Clearly the vertex set of Γ_{18}(Z) contains maximal ideals (2) and (3), the ideal vertex (9) is not maximal ideal and |Γ_x(R) | = 4>1.

The next result demonstrates that any two comaximal nontrivial ideals of R are adjacent in Γ_x(R).

Proposition 2.9: Any two comaximal nontrivial ideals of R are adjacent in Γ_x(R).

Proof: Let I and J are comaximal ideals of R. This means that I+J=R. Then there exist a∈I and b ∈J such that a + b =1. This implies that x=x(a+b)=xa+xb. From Lemma2.2, x ∈I∩J. This implies that xa, xb∈ I J . It follows that x∈ I J. This means that I and J are adjacent ideal vertices in Γ_x(R).

Example 11: Let Z be the ring of integers. Clearly Z= (2) + (3), the maximal ideal (2) is a vertex of Γ_8(Z) and $|\Gamma_8(\mathrm{Z})| = 2>1$.

The next result illustrates the adjacency of maximal ideals of R in the graph Γ_x(R).

Proposition 2.10: Let Γ_x(R) be non-empty graph**.** Then every two distinct maximal ideals of R, are adjacent in Γ_x(R).

Proof: Let M and N be two distinct maximal ideals of R. Clearly the ideal M+N contains both of M and N. Then either (M =M+N or N =M+N) or R=M+N. If M=M+N, then M⊂N, which is impossible because M is a maximal ideal. Therefore M≠M+N. By the same way we can show that N≠M+N. Thus R=M+N.This means that M and N are comaximal ideals. By Proposition2.9, M and N are adjacent ideal vertices in Γ_x(R).

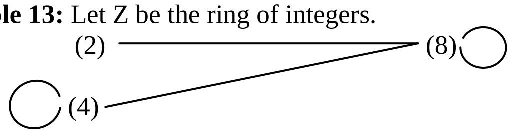
Example 12: Let Z be the ring of integers Z. Clearly (2) and (3) are maximal ideals of Z which are adjacent in Γ_{12}(Z).

We next turn to give the following result.

Theorem 2.11: If $\Gamma_x(R)$ is a planar graph, then R has at most four maximal ideals.

Proof: Suppose that R has five maximal ideals say M_1, M_2, M_3, M_4, M_5. By Proposition2.10, any two of M_1, M_2, M_3, M_4, M_5 are adjacent ideal vertices in $\Gamma_x(R)$. This means that $\Gamma_x(R)$ contains the graph K_5. Then by Theorem1.3, the graph $\Gamma_x(R)$ is not planar.This is contradiction that $\Gamma_x(R)$ is a planar graph. Therefore R has at most four maximal ideals.

Example 13: Let Z be the ring of integers.



Clearly $\Gamma_{16}(Z)$ is a planar graph and the only maximal ideal of Z_{16} is (8).

The next result shows that the principal ideal generated by x is not a vertex of Γ_x(R), if R is an integral domain.

Proposition 2.12: If R is an integral domain, then (x) is not an ideal vertex of Γ_x(R).

Proof: Suppose that (x) is an ideal vertex of Γ_x(R). Then there exists a nontrivial ideal I of R such that x ∈(x)·I. It follows that x=$\sum_i r_i x a_i$=x$\sum_i r_i a_i$ for some r_i ∈R and a_i ∈I. Since R is an integral domain, then the cancellation law gives us 1=$\sum_i r_i a_i$.This implies that 1∈I. This contradicts the fact that I is a nontrivial ideal of R. Therefore (x) is not an ideal vertex of Γ_x(R).

Example 14: Let Z be the ring of integers. Obviously, there is no ideal (x) of Z such that 6∈ (6) (x), therefore (6) is not a vertex of Γ_6(Z).

The ideal (x) may be an ideal vertex of Γ_x(R) if R is not an integral domain. The next example illustrates this.

Example 15: Let Z_{12} be the ring of integers modulo 12. Obviously Z_{12} is not integral domain and 6 ∈(6) (3). Hence (6) is an ideal vertex of $\Gamma_6(Z_{12})$.

We next give the following easy result.

Proposition 2.13: If x is an invertible element of R, then Γ_x(R)= $\emptyset$.

Proof: The prove is trivial.

Example 16: Let Z_8 be the ring of integers modulo 8. Since 3·3=1, then 3 is an invertible element of Z_8. On the other hand there are no ideals I and J of Z_8 such that $3 \in I \cdot J$. This means that $\Gamma_3(\mathrm{R}) = \emptyset$.

The converse of Propostion2.13 may not be true in general. We show this by the following example.

Example 17: Let Z be the ring of integers. Clearly there are no ideals I and J of Z such that 2$\in I \cdot J$. Then Γ_2(R)= $\emptyset$, while 2 is not invertible element in Z.

The next result illustrates the inclusive relation between two element ideal graphs of the same ring.

Proposition 2.14: If a is a factor of x, then Γ_a(R) is a subgraph of Γ_x(R).

Proof: Let I— J be an edge in Γ_a(R).This means that $\in I\,J$. Since a is a factor of x , then there exists b ∈R such that x=ab. Since $I\,J$ is an ideal of R, then x=ab∈ $I\,J$. This implies that I— J is an edge in Γ_x(R). Thus Γ_a(R) is a subgraph of Γ_x(R).

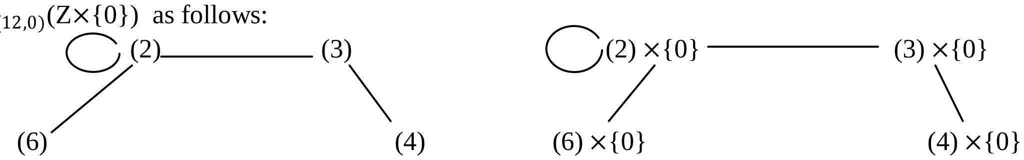
Example 18: Let Z be the ring of integers. Obviously 4 is a factor of 8 and Γ_4(Z) is a subgraph of Γ_8(Z).

The next result illustrates that the element ideal graph of two isomorphic rings R and S are also isomorphic.

Theorem 2.15: Let R and S be two rings such that f: R→S is a ring isomorphism. Then for any x∈ R the element ideal graphs Γ_x(R) and $\Gamma_{f(x)}$(S) are isomorphic.

Proof: Let V(Γ_x(R)) and V($\Gamma_{f(x)}$(S)) be the vertex set of Γ_x(R) and $\Gamma_{f(x)}$(R) respectively. We define the restricted function g:V(Γ_x(R))→ V($\Gamma_{f(x)}$(S)) of f by g(I)=f(I) for every vertex I of Γ_x(R). Since f is a bijective function, then g is also a bijective function. Suppose that I—J be an edge in Γ_x(R).Then x∈I J. This implies that f(x)∈ f(I J). Since f is a ring homomorphism , then f(IJ)=f(I)f(J)=g(I)g(J). This yields f(x)∈ g(I)g(J). This means that g(I) —g(J) is an edge in $\Gamma_{f(x)}$(S). Hence g preserves the adjacency property. Thus Γ_x(R) and $\Gamma_{f(x)}$(S) are isomorphic graphs.

Example 19: Let Z be the ring of integers. It is easy to show that the function f:Z→Z $\times\{0\}$ defined by f(x)=(x,0) is a ring isomorphism , where (a, b)+(c, d)= (a+c, b+d) and (a, b)·(c,d)=(ac+ad+bc,bd). Now we draw the graphs Γ_{12}(Z) and $\Gamma_{(12,0)}$(Z$\times\{0\}$) as follows:



Γ_{12}(Z) $\Gamma_{(12,0)}$(Z$\times\{0\}$)

Obviously the graphs Γ_{12}(Z) and $\Gamma_{(12,0)}$(Z$\times\{0\}$) are isomorphic.

Observe that the element ideal graphs of two rings are isomorphic, while the rings are not isomorphic. We illustrate this by the following example.

Example 20: Let Z_6 and Z_8 be the rings of integers modulo 6 an 8 respectively. Clearly the graphs Γ_4(Z_6) and Γ_4(Z_8) consist of the only loop (2) —(2). Then Γ_4(Z_6) and Γ_4(Z_8) are isomorphic, while Z_6 and Z_8 are not isomorphic.

3. CONNECTEDNESS AND COMPLETENESS OF ELEMENT IDEAL GRAPH

In this section we investigate the connectedness, completeness, planarity, bipartite and clique number of the element ideal graph.

The next result illustrates the connectedness and the diameter of the element ideal graph.

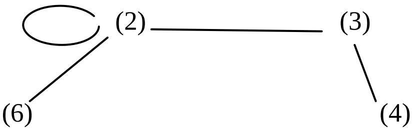
Theorem 3.1: The element ideal graph is connected and its diameter is less than or equal to 4.

Proof: Let I and J be any two distinct ideal vertices of the element ideal graph Γ_x(R).Then there exist ideal vertices K and L of Γ_x(R) which are adjacent to I and J respectively. Consider the ideal I+J. Now, we have two cases for I+J.

Case (1): If I and J are comaximal ideals, then by Proposition2.9, I and J are adjacent ideal vertices in Γ_x(R).

Case (2): Let I+J≠R. In this case I+J is a nontrivial ideal. Since I and J are contained in I + J , then by Proposition2.5, K and L are adjacent to I+J. Thus I — K — I+J — L—J is a path from I to J in Γ_x(R) (The ideal I+J may be equal to one of the ideals I , J , K and L). From both cases we have shown that there is a path between every two distinct vertices of $\Gamma_x(R)$. Hence $\Gamma_x(R)$ is connected. Obviously, from above cases we see that the diameter of $\Gamma_x(R)$ less than or equal to 4.

Example 21: Let Z be the ring of integers.



Clearly $\Gamma_{12}(Z)$ is a connected graph and its diameter is less than or equal to 4.

The next result shows that there is no element ideal star graph of order greater than two.

Theorem 3.2: If G is a star graph of order greater than 2, then G cannot be realized as an element ideal graph.

Proof: Suppose that $G=\Gamma_x(R)$ for some ring R and some $x\in R$. Let I be the center of G. Since G is a star graph of order greater than 2, then there exist two distinct vertices J, I and $K\neq I$ adjacent to I. Sine $I\subseteq I+K$, then by Proposition 2.5, I+K adjacent to J in G. But J is an end vertex of G, so I+K=I. It follows that $K\subseteq I$. This implies that $x\in IK\subseteq I^2$. This contradicts the fact that G has no any loop. Therefore G cannot be realized as an element ideal graph.

The element ideal graph could be a star graph if its order equal to 2. We illustrate it by the following example.

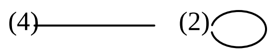
Example 22: Let Z be the ring of integers. Obviously the graph $\Gamma_{15}(Z)$ is a star which consists of the only edge (3)—(5).

The next result considers the completeness of the element ideal graph $\Gamma_x(R)$ under certain conditions.

Proposition 3.3: If x is an idempotent element of R, then $\Gamma_x(R)$ is a complete graph.

Proof: Let I and J be any two distinct ideal vertices of $\Gamma_x(R)$. By Lemma 2.2, $x\in I\cap J$. It follows that $x=xx\in I\cdot J$. This means that I and J are adjacent ideal vertices in $\Gamma_x(R)$. Therefore $\Gamma_x(R)$ is a complete graph.

Example 23: Let Z_{12} be the ring of integers modulo 12.



Obviously 4 is an idempotent element of Z_{12} and $\Gamma_4(Z_{12})$ is a complete graph.

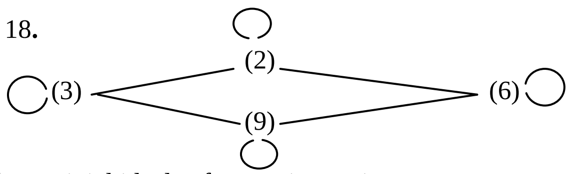
The next example shows that the converse of Proposition3.3 may not be true in general.

Example 24: Let Z be the ring of integers. Obviously the graph $\Gamma_6(Z)$ is complete which consists of the only edge (3) — (2), while 6 is not an idempotent element.

Corollary 3.4: If x=0, then $\Gamma_x(R)$ is a complete graph and all nontrivial ideals of R are its vertices.

Proof: The prove is trivial.

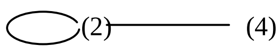
Example 25: Let Z_{18} be the ring of integers modulo 18**.**



Clearly the graph $\Gamma_0(Z_{18})$ is a complete graph and all nontrivial ideals of R are its vertices.

The converse of Corollary3.4 may not be true in general. We illustrate it by the following example.

Example 26: Let Z be the ring of integers.



Clearly the graph $\Gamma_8(Z)$ is a complete graph, while $8\neq 0$.

The following result gives a sufficient condition for the converse of Corollary3.4 to be true.

Proposition 3.5: Let R be a ring of zero Jacobson radical. If $\Gamma_x(R)$ is a complete graph whose vertices are all nontrivial ideals of R, then x=0.

Proof: Since $\Gamma_x(R)$ be a complete graph and all nontrivial ideals of R are its vertices. By Lemma 2.2, all nontrivial ideals of R contain x. This implies that $x\in J(R)$. Since R has zero Jacobson radical, then x=0.

Example 27: Let Z_6 be the ring of integers modulo 6. Obviously the ideals of Z_6 are (2) and (3) and the jacobson of R is J(R) $=\bigcap\{M: M \text{ is a maximal ideals of } M\}$ = (2)∩(3)=(0). The graph $\Gamma_0(Z_6)$ has the only verteces (2) and (3). Thus $\Gamma_0(Z_6)$ is a complete graph.

The next result demonstrates the completeness of $\bigcup_{x\neq 0}\Gamma_x$(R) under a condition on R.

Proposition 3.6: If R is an integral domain, then $\bigcup_{x\neq 0}\Gamma_x$(R) is a complete graph.

Proof: Let I and J be any two ideal vertices of $\bigcup_{x\neq 0}\Gamma_x$(R). Since R is an integral domain, then IJ$\neq (0)$.This means that IJ contains a non-zero element say x. So I— J is an edge in $\bigcup_{x\neq 0}\Gamma_x$(R).Thus $\bigcup_{x\neq 0}\Gamma_x$(R) is a complete graph.

Example 28: Let Z be the ring of integers. Obviously Z is an integral domain and $\bigcup_{x\neq 0}\Gamma_x$(Z) is a complete graph.

The next example illustrates that the graph $\bigcup_{x\neq 0}\Gamma_x$(R) may not be a complete graph in general.

Example 29: Let Z_{12} be the ring of integers modulo 12. Since 8∈(2)(4) and 6∈ (2)(3), then (3) and (4) are two ideal vertices of $\bigcup_{x\neq 0}\Gamma_x(Z_{12})$.On the other hand (3)(4)=(0).This means that (3) and (4) are not adjacent in $\bigcup_{x\neq 0}\Gamma_x(Z_{12})$. Hence $\bigcup_{x\neq 0}\Gamma_x$(R) is not a complete graph.

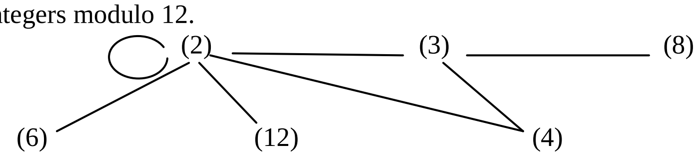
The next result limits the girth of the element ideal graph under the conditions on an its edge.

Proposition 3.7: Let x be a non-zero element of R. If Γ_x(R) contains an edge I—J such that:
1. I and J are not comaximal ideals of R.
2. Any one of ideals I and J does not contains the other.

Then the girth of Γ_x(R) is equal to three.

Proof: Let I—J be an edge of Γ_x(R) such that I and J are not comaximal with I⊈J and J⊈I. Clearly I+J≠R. Since I.J⊆I+J, then I+J$\neq (0)$. By Proposition 2.5, I+J is adjacent to both I and J. Since niether I⊆J nor J⊆I ,then neither I+J=I nor I+J=J. This means that I, J and I+J are distinct vertices of Γ_x(R). Hence I+J—I—J—I+J is a cycle in Γ_x(R). Thus the girth of Γ_x(R) is equal to three.

Example 30: Let Z_{24} be the ring of integers modulo 12.



Since (3) + (4) ≠Z, then (3) and (4) are not comaximal ideals of Z, which are adjacent ideal vertices in Γ_{24}(Z) and the girth of Γ_{24}(Z) is equal to 3.

The next result determines the lower bounds for the clique number of the graph $\bigcup_{x\neq 0}\Gamma_x$(R).

Theorem 3.8: Let a∈R\{0}, and let n>6 be the smallest positive integer such that $a^n = 0$. Then cl($\bigcup_{x\neq 0}\Gamma_x$(R))=$\begin{cases}\frac{n}{2}-1 & n\in Z_e\\ \frac{n-1}{2} & n\in Z_o\end{cases}$, where Z_e and Z_o are the set of even and odd integers respectively.

Proof: Since n is a smallest integer such that $a^n = 0$, then the principal ideal (a^i) is non zero ideal for every i=1,2,…,n-1. On the other hand a^i is a zero divisor of R for every i=1,2,…,n-1, so $(a^i)\neq R$. Therefore (a^i) is a nontrivial ideal of R. Now if n is an even number , then (a^1) , (a^2),…, $(a^{\frac{n}{2}-1})$ are adjacent ideal vertices in $\bigcup_{x\neq 0}\Gamma_x$(R). This means that the graph $\bigcup_{x\neq 0}\Gamma_x$(R) contains a complete subgraph of vertices (a^1),(a^2),…, $(a^{\frac{n}{2}-1})$. Thus the clique number of $\bigcup_{x\neq 0}\Gamma_x$(R) is greater than or equal to $\frac{n}{2}-1$. If n is an odd number, then (a^1),(a^2),…, $(a^{\frac{n-1}{2}})$ are adjacent ideal vertices in $\bigcup_{x\neq 0}\Gamma_x$(R). This means that the graph $\bigcup_{x\neq 0}\Gamma_x$(R) contains a complete subgraph of ideal vertices (a^1),(a^2),…, $(a^{\frac{n-1}{2}})$. Thus the clique number of $\bigcup_{x\neq 0}\Gamma_x$(R) is greater than or equal to $\frac{n-1}{2}$.

Example 31: Let Z_{128} be the ring of integers modulo 128. Clearly 2 is an element of Z_{128} and n=7 is the smallest odd integer in which 2^7=0. Since any two of (2^1),(2^2) and (2^3) are adjacent ideal vertices in $\bigcup_{x\neq 0}\Gamma_x(Z_{128})$, then the graph $\bigcup_{x\neq 0}\Gamma_x(Z_{128})$ contains K_3 . Thus cl($\bigcup_{x\neq 0}\Gamma_x(Z_{128})$) ≥ $\frac{7-1}{2}$=3.

The next result considers the relationship between two of the element ideal graph.

Theorem 3.9: Let Γ_x(R) and Γ_y(R) be nonempty graphs. Then Γ_x(R) + Γ_y(R) is a subgraph of Γ_{xy}(R).

Proof: Let I— J be an edge in Γ_x(R) + Γ_y(R).We have the following cases:

Case (1): Both I and J are ideal vertices of Γ_x(R).This means that either $x \in$ I J. Since I J is an ideal of R, then x$y \in$ I J . Thus I— J is an edge in Γ_{xy} (R).

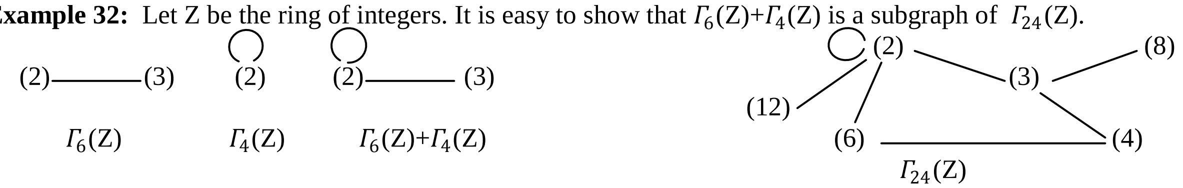
Case (2): Both I and J are ideal vertices of Γ_y(R). By the same way of case (1) we obtain that I— J is an edge inΓ_{xy} (R).

Case (3): The ideal I is an ideal vertex of Γ_x(R) or Γ_y(R) and J is an ideal vertex of the other graph. Suppose that I and J are ideal vertices of Γ_x(R) and Γ_y(R) respectively. It follows from Lemma2.2 that x$\in$ I and $y \in$ J, and hence x$y \in$ I J. This means that I— J is an edge in Γ_{xy} (R).

From each case we obtain that Γ_x(R)+Γ_y(R)$\subseteq\Gamma_{xy}$ (R).

The next example illustrates that Γ_{xy} (R) may not be a subgraph of Γ_x(R) + Γ_y(R).

Example 32: Let Z be the ring of integers. It is easy to show that Γ_6(Z)+Γ_4(Z) is a subgraph of Γ_{24}(Z).



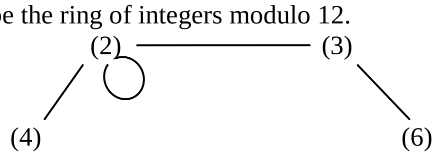
Clearly (4) —(6) is an edge in Γ_{24}(Z) , while (4) —(6) is not an edge in Γ_6(Z)+Γ_4(Z).

The next result illustrates that the graph $\bigcup_{x\neq 0}\Gamma_x$(R) is not a complete bipartite.

Theorem 3.10: If all nontrivial ideals of R are vertices of $\bigcup_{x\neq 0}\Gamma_x$(R), then $\bigcup_{x\neq 0}\Gamma_x$ (R) $\neq K_{mn}$ for every integers m, n>1.

Proof: Suppose that $\bigcup_{x\neq 0}\Gamma_x$ (R) $= K_{mn}$ for some integers m, n>1, with the partite A and B. Clearly, all vertices of A are adjacent in AG(R) and all vertices of B are adjacent in AG(R). So AG(R) consists of two subgraph and there is no edge between any two vertices I in the first graph and J in the second graph. This contradicts Theorem1.5, because AG(R) is connected graph. Therefore $\bigcup_{x\neq 0}\Gamma_x$ (R) $\neq K_{mn}$ for every integers m, n>1.

Example 33: Let Z_{12} be the ring of integers modulo 12.



Clearly $\bigcup_{x\neq 0}\Gamma_x$ (Z_{12}) is not a complete bipartite graph.

4. THE GRAPHS Γ_x(R) and AG(R)

In this section we consider the relationship between Γ_x(R) and AG(R).

We start this section with the following result.

Theorem4.1: Let Γ_x(R) and $AG(R)$ are non-empty graphs**.** Then $E(\Gamma_x$(R))$\cap$ $E(AG(R)) \neq \emptyset$ if and only if x=0.

Proof: Let x=0. Since $AG(R)$ is a non-empty graph, there exist two ideals of R which are adjacent in $AG(R)$**.** Since 0$\in IJ$, then I and J are adjacent in Γ_0(R). Hence $E(\Gamma_x$(R))$\cap$ $E(AG(R)) \neq \emptyset$.

Conversely, if $E(\Gamma_x$(R))$\cap$ $E(AG(R)) \neq \emptyset$, then there exists two distinct ideals I and J of R such that I— J is an edge of $E(\Gamma_x$(R))$\cap$ $E(AG(R))$.This means that $x \in IJ$ and I J=(0). This gives us x=0.

The following result illustrates the relation between Γ(R) and Γ_x(R).

Corollary 4.2: Let $|\Gamma_x$(R)$|$>1. If Γ(R) is a complete graph, then Γ_x(R) is also a complete graph.

Proof: Since $|\Gamma_x(R)|>1$, the graph $\Gamma_x(R)$ contains an edge say I— J. Since Γ(R) is a complete graph, then the graph AG(R) is also a complete graph. By Theorem1.4, the vertex set of AG(R) and the set of nonzero proper ideals of R have the same cardinality. So I and J are also ideal vertices of AG(R). It follows from the completeness of AG(R) that I— J is an edge of AG(R). This means that $E(\Gamma_x(R)) \cap E(AG(R)) \neq \emptyset$. Then by Theorem 4.1, x=0. It follows from Corollary 3.4 that $\Gamma_x(R)$ is a complete graph.

Example 34: Let Z_9 be the ring of integers modulo 9. Clearly $\Gamma(Z_9)$ is a complete graph and $\Gamma_x(Z_9)$ is a trivial graph which consists of the loop (3) — (3).

The following result illustrates that the element ideal graph $\Gamma_0(R)$ consists of the edge in the annihilating ideal graph and the element ideal graph $\Gamma_x(R)$ for every non zero element x.

Proposition 4.3: For any ring R, $\Gamma_0(R) = AG(R) \cup (\bigcup_{x\neq 0} \Gamma_x(R))$.

Proof: By Corollary3.4, $\Gamma_0(R)$ is a complete graph whose vertices are all nontrivial ideals of R. Then $AG(R) \cup (\bigcup_{x\neq 0} \Gamma_x(R)) \subseteq \Gamma_0(R)$. Now suppose that I— J is an edge of $\Gamma_0(R)$. Clearly either I J=(0) or I J≠(0). This means that either I— J is an edge in $AG(R)$ or I— J is an edge in $\Gamma_x(R)$ for some $x \neq 0$. This implies that I— J is an edge in $AG(R) \cup (\bigcup_{x\neq 0} \Gamma_x(R))$. Hence $\Gamma_0(R) \subseteq AG(R) \cup (\bigcup_{x\neq 0} \Gamma_x(R))$. Thus $\Gamma_0(R) = AG(R) \cup (\bigcup_{x\neq 0} \Gamma_x(R))$.

Example 35: Let Z_6 be the ring of integers modulo 6. It is easy to show that $\Gamma_1(Z_6)=\Gamma_2(Z_6)= \Gamma_5(Z_6)= \emptyset$, the graph $\Gamma_3(Z_6)$ consists of (3) —(3) , the graph $\Gamma_4(Z_6)$ consists of (2) —(2) and the graph $AG(Z_6)$ consists of (2) —(3).

Now $AG(Z_6) \cup (\bigcup_{x=1}^{5} \Gamma_x(Z_6)) = AG(Z_6) \cup \Gamma_3(Z6) \cup \Gamma_4(Z_6)$ consists of the edge (2) —(3) and two loops (2) —(2) and (3) —(3), which represents the graph $\Gamma_0(Z_6)$.

The next result illustrates that x is a zero divisor of R if the vertex set of $\Gamma_x(R)$ intersects the vertex set of AG(R).

Theorem 4.4: Let R be a non-domain. If the vertex sets of $\Gamma_x(R)$ and AG(R) have a common vertex, then $x \in Z(R)$.

Proof: Suppose that the vertex set of $\Gamma_x(R)$ and the vertex set of AG(R) have a common vertex say I. Now the ideal vertex I of AG(R) gives us I⊆Z(R). From Lemma2.2 , the ideal vertex I of $\Gamma_x(R)$ gives us $x \in I$. Thus x∈Z(R).

Example 36: Let Z_{12} be the ring of integers modulo 12. Clearly (2) (6) = (0) and 8∈(2)(4). Then (2) is an ideal vertex of both AG(R) and $\Gamma_{12}(Z_{12})$, in the same time 2 is a zero divisor of Z_{12}**.**

Corollary 4.5: Let R be a non-domain.If $\Gamma_x(R)$ has a vertex which is a nilpotent ideal of R, then x is a zero divisor of R.

Proof: Let I be a nilpotent ideal vertex of $\Gamma_x(R)$. Sine I is a nilpotent ideal, then there exists a smallest integer m>1 such that $I\, I^{m-1}=I^m=(0)$. This means that I is an ideal vertex of AG(R).Since I is an ideal vertex of $\Gamma_x(R)$, then Theorem4.4 gives us x is a zero divisor of R.

Example 37: Let Z_8 be the ring of integers modulo 8. Clearly (2) is a nilpotent ideal of Z_8 and 2 is a zero divisor of Z_8.

The cardinality of the element ideal graph may be less than the cardinality of the set of nonzero proper ideals of R. We illustrate it by the following example.

Example 38: Let Z_8 be the ring of integers modulo 8. Since the vertex set of $\Gamma_x(Z_8)$ is {(2)}, then the cardinality of $\Gamma_x(Z_8)$ is equal to 1, and The cardinality of the set of nonzero proper ideals {(2), (4)} of Z_8 is equal to 2. So the cardinality of $\Gamma_x(Z_8)$ is less than the cardinality of the set of nonzero proper ideals of Z_8.

The next result demonstrates the relation between the cardinality of the element ideal graph and the annihilating ideal graph.

Proposition 4.6: If R is a non-domain, then the cardinality of $\Gamma_x(R)$ is less than or equal to the cardinality of AG(R) for every $x \in R$.

Proof: From Theorem1.4, the set of vertices of AG(R) and the set of nonzero proper ideals of R have the same cardinality. Thus the cardinality of $\Gamma_x(R)$ is less than or equal to the cardinality of AG(R) for every $x \in R$.

Example 39: Let Z_8 be the ring of integers modulo 8. It is easy to show that the cardinality of $AG(Z_8)$ is equal to 2 and the cardinality of $\Gamma_4(Z_8)$ is equal to 1.

If R is an integral domain, then the cardinality of AG(R) is zero. Hence the cardinality of $\Gamma_x(R)$ is greater than or equal to the cardinality of AG(R) for every $x \in R$.We illustrate it in the following example.

Example 40: Let Z be the ring of integers. Clearly $\text{AG(Z)} = \emptyset$ and the graph $\Gamma_{12}(Z)$ has four vertices. So the cardinality of $\Gamma_x(Z)$ is greater than or equal to the cardinality of AG(Z).

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