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FIXED POINT THEOREMS FOR WEAKLY CONTRACTIVE TYPE SELF MAPS ON A PARTIALLY ORDERED SET WITH A METRIC

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ABSTRACT

In this paper, we prove fixed point theorems (existence and uniqueness) for weakly contractive type self maps on a partially ordered set with a metric. An example is given to show the existence of more than one fixed point if certain condition is violated. We observe that the fixed point set gives rise to a partition of partially ordered set.

Key words: *ordered metric space, rational expression, partially ordered set, weakly contractive map, weakly contractive type map.*

Mathematical subject classification (2010): *47H10, 54H25.*

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1. INTRODUCTION AND PRELIMINARIES

Banach contraction principle was generalized by many authors in several ways. One among the ways of generalizations of Banach contraction mapping theorem was replacing the right hand side of the inequality by the related terms involving rational expressions.

In 1980, Jaggi and Dass [13] proved the existence of fixed point for a self map T on a complete metric space using rational expression.

Theorem: 1.1 (Jaggi and Dass [13]) Let T be a self map of a metric space (X, d) which satisfy

(i) for some $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$ such that

$$d(Tx, Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,Ty)+d(y,Tx)+d(x,y)} + \beta d(x,y) \tag{1.1.1}$$

for all $x, y \in X, x \neq y$

(ii) there exists $x_0 \in X$ such that $\{T^n x_0\}$ has a convergent subsequence with limit z in X. Then z is the unique fixed point of T in X.

Further, Alber and Guerre-Delabrire [3] defined weakly contractive map on a Hilbert space and established a fixed point theorem for such map. This theorem was extended to metric spaces by Rhoades [22]. Existence of fixed points of self maps satisfying weakly contractive type inequalities have been studied by several authors. For example we refer [3, 5, 6, 11, 12, 14, 22].

Definition: 1.2 [22] Let (X, d) be a complete metric space. A mapping $T: X \to X$ is said to be a weakly contractive map if $d(Tx, Ty) \leq d(x, y) - \varphi\big(d(x, y)\big)$ for all $x, y \in X$ where
$\varphi: [0, \infty) \to [0, \infty)$ is continuous and non decreasing function with $\varphi(t) = 0$ iff $t = 0$.

Theorem: 1.3 [22] Let (X, d) be a complete metric space and T be a weakly contractive map on$X$. Then T has a unique fixed point.

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Now a days a lot of research work is going on in the field of existence of fixed points in partially ordered metric spaces. The first result in this direction was given by Ran and Reurings [20]. Many other results on the existence of fixed points or common fixed points in ordered spaces were studied by several authors. For more literature on the existence of fixed points in ordered metric spaces we refer [1, 2, 4, 7, 8, 9, 10, 12, 14, 15, 16, 17, 18, 20, 23].

Harjani and Sadarangani 12] proved the following fixed point theorem for weak contractive maps in the context of partially ordered metric spaces.

Theorem: 1.4 [12] Let $(X, \leq)$ be a poset and suppose that there exists a metric d on X such that (X, d) is complete metric space. Suppose $T: X \to X$ is a non decreasing function satisfying the following:

(1) $\varphi: [0, \infty) \to [0, \infty)$ is continuous, non decreasing function such that $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$ such that $d(Tx, Ty) \leq d(x, y) - \varphi\big(d(x, y)\big)$ for all $x, y \in X, x \geq y$
(2) either T is continuous or
(3) X has the following property:

If $\{x_n\}$ is a non decreasing sequence with $x_n \to x$, then $x_n \leq x \;\forall\, n \geq 1$.

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$.

Then T has a fixed point.

We introduce the concept of a weakly contractive type map as follows:

Definition: 1.5 Let (X, d) be a metric space. A self map $T: X \to X$ is said to be a weakly contractive type map if there exists a function $\varphi: [0, \infty) \to [0, \infty)$ which is increasing and $\varphi(t) > 0$ for $t > 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi\big(d(x, y)\big)$$

(1) The purpose of this paper is to prove the existence and uniqueness of fixed points for a weakly contractive type map in partially ordered sets with a metric, by eliminating the continuity condition on φ and also the condition $\lim_{t\to\infty} \varphi(t) = \infty$ in Theorem 1.4.

3. MAIN RESULTS

First we state the following lemma, whose proof can be easily established.

Lemma: 2.1 Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X such that $\{d(x_n, x_{n+1})\}$ is decreasing sequence which decreases to 0. Suppose $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $m(k) > n(k) > k$ satisfying the following property:

(1) $d\big(x_{m(k)}, x_{n(k)}\big) \geq \varepsilon$ and $d\big(x_{n(k)}, x_{m(k)-1}\big) < \varepsilon$
Consequently, we get
(2) $\lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)}\big) = \varepsilon$
(3) $\lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)-1}\big) = \varepsilon$
(4) $\lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)-1}\big) = \varepsilon$
(5) $\lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)}\big) = \varepsilon$

Theorem: 2.2 Let $(X, \leq)$ be a set. Suppose that there exists a metric d on X such that (X, d) is a metric space. Suppose $T: X \to X$ is a non decreasing mapping satisfying the following inequality : there exists $\varphi: [0, \infty) \to [0, \infty)$ which is increasing and $\varphi(t) > 0$ if $t > 0$ and

$$d(Tx, Ty) \leq d(x, y) - \varphi\big(d(x, y)\big) \tag{2.2.1}$$

for all $x, y \in X$, whenever x, y are comparable

Suppose there exists $x_0 \in X$ such that $x_0 \leq Tx_0$. Write $x_{n+1} = Tx_n \;\forall\, n = 0,1,2, \ldots$

Then $\{x_n\}$ is a Cauchy sequence.

Proof: First we observe that $\varphi(0) = 0$, since by taking $x = y$ in (2.2.1), we get
$d(Tx, Tx) \leq d(x, x) - \varphi\big(d(x, x)\big)$

$\Rightarrow 0 \leq 0 - \varphi(0) \Rightarrow \varphi(0) \leq 0 \Rightarrow \varphi(0) = 0.$

Now, we show that $\{x_n\}$ is a non decreasing sequence. i.e $x_n \leq x_{n+1}\ \forall\ n \geq 0$ (2.2.2)

We have $x_0 \leq Tx_0 = x_1$.

Therefore (2.2.2) is true for $n = 0$.

Suppose that (2.2.2) is true for some $n = m$. i.e. $x_m \leq x_{m+1}$.

We have $Tx_m \leq Tx_{m+1}$ ($\because T$ is increasing)

$\Rightarrow x_{m+1} \leq x_{m+2}$

Therefore (2.2.2) is true for all $n = 0, 1, 2, \ldots$

Therefore $\{x_n\}$ is a non decreasing sequence.

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n = Tx_n = x_{n+1}$.

Hence $x_{n+2} = Tx_{n+1} = Tx_n = x_n$.

Then $x_n = x_{n+1} = x_{n+2} = \cdots$

Hence $\{x_n\}$ is a Cauchy sequence.

Suppose $x_n \neq x_{n+1}\ \forall\ n$.

Since $x_n \geq x_{n-1}\ \forall\ n \geq 1$ from (2.2.1), we have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq d(x_n, x_{n-1}) - \varphi\big(d(x_n, x_{n-1})\big) < d(x_n, x_{n-1}) \tag{2.2.3}$$

$\therefore \{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence of positive numbers and hence converges, say, to $\delta \geq 0$.

Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = \delta$.

We shall now show that $\delta = 0$.

From (2.2.3), we have

$$\varphi(d(x_n, x_{n-1})) \leq d(x_n, x_{n-1}) - d(x_{n+1}, x_n) \to \delta - \delta = 0 \text{ as } n \to \infty$$

$\therefore\ \varphi(d(x_n, x_{n-1})) \to 0$ as $n \to \infty$.

But $\delta \leq d(x_n, x_{n+1})\ \forall\ n$

$\Rightarrow \varphi(\delta) \leq \varphi\big(d(x_n, x_{n+1})\big)$ ($\because \varphi$ is increasing)

$\Rightarrow \varphi(\delta) \leq \lim_{n\to\infty} \varphi\big(d(x_n, x_{n+1})\big) = 0$

$\therefore\ \varphi(\delta) = 0$, consequently $\delta = 0$. ($\because \delta > 0 \Rightarrow \varphi(\delta) > 0$)

i.e. $\lim_{n\to\infty} \varphi\big(d(x_n, x_{n+1})\big) = 0$ (2.2.4)

Now, we show that $\{x_n\}$ is a Cauchy sequence.

Otherwise, by Lemma 2.1, there exists an $\varepsilon > 0$ and sequences $\{m(k)\}$ *and* $\{n(k)\}$ with $m(k) > n(k) > k$ such that

$d\big(x_{m(k)}, x_{n(k)}\big) \geq \varepsilon$ and $d\big(x_{n(k)}, x_{m(k)-1}\big) < \varepsilon$.

Consequently
(1) $\lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)}\big) = \varepsilon$
(2) $\lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)-1}\big) = \varepsilon$
(3) $\lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)-1}\big) = \varepsilon$
(4) $\lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)}\big) = \varepsilon$

Since $m(k) > n(k)$, we have $x_{m(k)-1} < x_{n(k)-1}$. Hence

$$d\big(x_{m(k)}, x_{n(k)}\big) = d\big(Tx_{m(k)-1}, Tx_{n(k)-1}\big) \le d\big(x_{m(k)-1}, x_{n(k)-1}\big) - \varphi\left(d\big(x_{m(k)-1}, x_{n(k)-1}\big)\right)$$

$\Rightarrow \varphi\left(d\big(x_{m(k)-1}, x_{n(k)-1}\big)\right) \le d\big(x_{m(k)-1}, x_{n(k)-1}\big) - d\big(x_{m(k)}, x_{n(k)}\big) \to \varepsilon - \varepsilon = 0$ as $k \to \infty$

$\therefore \lim\limits_{n\to\infty} \varphi\left(d\big(x_{m(k)-1}, x_{n(k)-1}\big)\right) = 0$

From (2), we have $\left(d\big(x_{m(k)-1}, x_{n(k)-1}\big)\right) > \frac{\varepsilon}{2}$ for large k.

$\therefore \varphi\left(\frac{\varepsilon}{2}\right) \le \varphi\left(d\big(x_{m(k)-1}, x_{n(k)-1}\big)\right) \to 0$ as $k \to \infty$

Therefore $\varphi\left(\frac{\varepsilon}{2}\right) = 0$, a contradiction.

$\therefore \varepsilon = 0$, again a contradiction.

Therefore $\{x_n\}$ is a Cauchy sequence.

Theorem: 2.3 In addition to the hypothesis of Theorem 2.2, suppose (X, d) is complete and T is continuous. Then T has a fixed point.

Proof: Suppose $x_0 \le Tx_0$. Let $x_{n+1} = Tx_n, n = 0, 1, 2, \ldots$..

Then by (2.2.2), $\{x_n\}$ is an increasing sequence.

Now, by Theorem 2.2, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists y such that $\lim_{n\to\infty} x_n = y$.

Since T is continuous, $Tx_n \to Ty$ i.e $x_{n+1} \to Ty$

But $x_{n+1} \to y$.

Therefore $Ty = y$.

$\therefore y$ is a fixed point of T.

Theorem: 2.4 In addition to the hypothesis of Theorem 2.2, suppose (X, d) is complete and X has the following property:

If $\{z_n\}$ is an increasing sequence with $z_n \to z$, then $z \le z_n \ \forall\ n$ (2.4.1)

Then T has a fixed point.

Proof: Suppose $x_0 \le Tx_0$. Let $x_{n+1} = Tx_n \ \forall\ n = 0, 1, 2, \ldots$..

Then by (2.2.2), $\{x_n\}$ is an increasing sequence.

Now, by Theorem 2.2, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists y such that $\lim_{n\to\infty} x_n = y$.

We may assume that $x_n \neq y \ \forall\, n$.

By (2.4.1), $y \geq x_n \ \forall\, n$

$\therefore$ By (2.2.1), we have

$$d(x_{n+1}, Ty) = d(Tx_n, Ty) \leq d(x_n, y) - \varphi\big(d(x_n, y)\big) \leq d(x_n, y) \to 0 \text{ as } n \to \infty. \tag{2.4.2}$$

Therefore $d(x_{n+1}, Ty) \to 0$ as $n \to \infty$

But $d(x_{n+1}, Ty) \to d(y, Ty)$ $\quad$ ($\because x_{n+1} \to y$)

$\therefore\ d(y, Ty) = 0$

$\therefore Ty = y$

Therefore y is a fixed point of T.

Combining Theorem 2.2, Theorem 2.3 and Theorem 2.4, we have the following theorem.

Theorem: 2.5 Let $(X, \leq)$ be a poset and suppose that there exists a metric d on X such that (X, d) is complete. Suppose $T: X \to X$ is a non decreasing function satisfying the following:
(1) $\varphi: [0, \infty) \to [0, \infty)$ is a non decreasing function such that $\varphi(t) > 0$ if $t > 0$
(2) $d(Tx, Ty) \leq d(x, y) - \varphi\big(d(x, y)\big)$ for all $x, y \in X$ whenever x, y are comparable
(1) there exists $x_0 \in X$ such that $x_0 \leq Tx_0$
(2) either T is continuous or X has the following property: If $\{x_n\}$ is a non decreasing sequence with $x_n \to x$, then $x_n \leq x \ \forall\, n \geq 1$.

Then T has a fixed point.

(In fact, the sequence $\{x_n\}$ denoted by $x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots$ with x_0 as in (3), is a Cauchy sequence and hence converges, say to y, which is a fixed point of T)

Corollary: 2.6 (Theorem, [12]) Let $(X, \leq)$ be a poset and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose $T: X \to X$ is a non decreasing function satisfying the following:
(4) $\varphi: [0, \infty) \to [0, \infty)$ which is continuous, non decreasing function such that $\varphi(t) > 0$ for $t > 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$
(5) $d(Tx, Ty) \leq d(x, y) - \varphi\big(d(x, y)\big)$ for all $x, y \in X, x \geq y$
(6) there exists $x_0 \in X$ such that $x_0 \leq Tx_0$
(7) either T is continuous or X has the following property: if $\{x_n\}$ is a non decreasing sequence with $x_n \to x$, then $x_n \leq x \ \forall\, n \geq 1$.
Then T has a fixed point.

Note: 2.7 We note that in Theorem 2.5, we have successfully discarded the two properties of φ mentioned in the Corollary 2.6, namely (1) φ is continuous (2) $\lim_{t \to \infty} \varphi(t) = \infty$.

Theorem: 2.8 Let $(X, \leq)$ be a poset. Suppose that there exists a metric d on X such that (X, d) is a metric space. Suppose $T: X \to X$ is a non decreasing mapping satisfying the following:
(1) $\varphi: [0, \infty) \to [0, \infty)$ is increasing and $\varphi(t) > 0$ if $t > 0$
(2) $d(Tx, Ty) \leq d(x, y) - \varphi\big(d(x, y)\big)$ for all $x, y \in X,$ whenever x, y are comparable
(3) there exists $x_0 \in X$ such that $x_0 \geq Tx_0$.

Then the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n \ \forall\, n = 0,1,2, \ldots$ is a Cauchy sequence.

Proof: Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n \ \forall\, n = 0,1,2, \ldots$

by induction, first we show that $x_n \geq x_{n+1} \ \forall\, n = 0, 1, 2, \ldots$ (2.8.1)

We have $x_0 \geq Tx_0 = x_1$.

Therefore (2.8.1) is true for $n = 0$.

Assume that (2.8.1) is true for some $n = m$. i.e. $x_m \geq x_{m+1}$.

Since T is increasing, we have $Tx_m \geq Tx_{m+1}$

$\Rightarrow x_{m+1} \geq x_{m+2}$

Therefore (2.8.1) is true for all $n \geq 0$.

Therefore $\{x_n\}$ is a decreasing sequence.

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n = Tx_n = x_{n+1}$.

Hence $x_{n+2} = Tx_{n+1} = x_n$.

Then $x_n = x_{n+1} = x_{n+2} = \cdots$

Hence $\{x_n\}$ is a Cauchy sequence.

Suppose $x_n \neq x_{n+1}\ \forall\ n$.

Since $x_n \leq x_{n-1}\ \forall\ n \geq 1$ from (2.2.1), we have

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) &\leq d(x_n, x_{n-1}) - \varphi\big(d(x_n, x_{n-1})\big) \\ &< d(x_n, x_{n-1}) \end{aligned} \qquad (2.8.2)$$

$\therefore$ $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of positive numbers and converges, say to $\delta \geq 0$.

Hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = \delta$.

We shall now show that $\delta = 0$.

From (2.2.3), we have $\varphi(d(x_n, x_{n-1})) \leq d(x_n, x_{n-1}) - d(x_{n+1}, x_n) \to \delta - \delta = 0$ as $n \to \infty$

$\therefore$ $\varphi(d(x_n, x_{n-1})) \to 0$ as $n \to \infty$

But $\delta \leq d(x_n, x_{n+1})\ \forall\ n$

$\Rightarrow \varphi(\delta) \leq \varphi\big(d(x_n, x_{n+1})\big)$ ($\because \varphi$ is increasing)

$\Rightarrow \varphi(\delta) \leq \lim_{n\to\infty} \varphi\big(d(x_n, x_{n+1})\big) = 0$

$\therefore$ $\varphi(\delta) = 0$, consequently $\delta = 0$. ($\because \delta > 0 \Rightarrow \varphi(\delta) > 0$)

i.e. $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ (2.8.3)

Now, we show that $\{x_n\}$ is a Cauchy sequence.

Otherwise, by Lemma 2.1, there exists an $\varepsilon > 0$ and sequences $\{m(k)\}$ *and* $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d\big(x_{m(k)}, x_{n(k)}\big) \geq \varepsilon$ and $d\big(x_{n(k)}, x_{m(k)-1}\big) < \varepsilon$.

Consequently
(1) $\lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)}\big) = \varepsilon$
(2) $\lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)-1}\big) = \varepsilon$
(3) $\lim_{k\to\infty} d\big(x_{m(k)}, x_{n(k)-1}\big) = \varepsilon$
(4) $\lim_{k\to\infty} d\big(x_{m(k)-1}, x_{n(k)}\big) = \varepsilon$

Since $m(k) > n(k)$, we have $x_{m(k)-1} < x_{n(k)-1}$, we have

$$d\big(x_{m(k)}, x_{n(k)}\big) = d\big(Tx_{m(k)-1}, Tx_{n(k)-1}\big) \leq d\big(x_{m(k)-1}, x_{n(k)-1}\big) - \varphi\left(d\big(x_{m(k)-1}, x_{n(k)-1}\big)\right)$$

$\Rightarrow \varphi\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right)\right) \leq d\left(x_{m(k)-1}, x_{n(k)-1}\right) - d\left(x_{m(k)}, x_{n(k)}\right) \to \varepsilon - \varepsilon = 0$ as $k \to \infty$

$\therefore\ \lim_{n\to\infty} \varphi\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right)\right) = 0$

From (2), we have $\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right)\right) > \frac{\varepsilon}{2}$ for large k.

$\therefore\ \varphi\left(\frac{\varepsilon}{2}\right) \leq \varphi\left(d\left(x_{m(k)-1}, x_{n(k)-1}\right)\right) \to 0$ as $k \to \infty$

Therefore $\varphi\left(\frac{\varepsilon}{2}\right) = 0$, a contradiction.

$\therefore\ \varepsilon = 0$, again a contradiction.

Therefore $\{x_n\}$ is a Cauchy sequence.

Theorem: 2.9 In addition to the hypothesis of Theorem 2.8, suppose (X, d) is complete and T is continuous. Then T has a fixed point.

Proof: Suppose $x_0 \geq Tx_0$. Let $x_{n+1} = Tx_n, n = 0, 1, 2, \ldots$..

Then by (2.8.1), $\{x_n\}$ is a decreasing sequence.

Now, by Theorem 2.8, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists y such that $\lim_{n\to\infty} x_n = y$.

Since T is continuous, $Tx_n \to Ty$ i.e $x_{n+1} \to Ty$

But $x_{n+1} \to y$.

Therefore $Ty = y$.

$\therefore$ y is a fixed point of T.

Theorem: 2.10 In addition to hypothesis of Theorem 2.8, suppose (X, d) is complete and X has the following property:

If $\{z_n\}$ is a decreasing sequence with $z_n \to z$, then $z_n \geq z\ \forall\ n$ (2.10.1)

Then T has a fixed point.

Proof: Suppose $x_0 \geq Tx_0$. Let $x_{n+1} = Tx_n\ \forall\ n = 0, 1, 2, \ldots$..

Then by (2.8.1), $\{x_n\}$ is a decreasing sequence.

Now, by Theorem 2.8, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists y such that $\lim_{n\to\infty} x_n = y$.

We may assume that $x_n \neq y\ \ \forall\, n$.

By (2.10.1), $x_n \geq y\ \forall\ n$

Then we have

$d(x_{n+1}, Ty) = d(Tx_n, Ty) \leq d(x_n, y) - \varphi\big(d(x_n, y)\big) < d(x_n, y) \to 0$ as $n \to \infty$. (2.10.2)

Therefore $d(x_{n+1}, Ty) \to 0$ as $n \to \infty$

But $d(x_{n+1}, Ty) \to d(y, Ty)$ $(\because x_{n+1} \to y)$

$\therefore\ d(y, Ty) = 0$

$\therefore Ty = y$

Therefore y is a fixed point of T.

Combining Theorem 2.8, Theorem 2.9 and Theorem 2.10, we have the following theorem.

Theorem: 2.11 Let $(X, \leq)$ be a poset and suppose that there exists a metric d on X such that (X, d) is complete metric space. Suppose $T: X \to X$ is a non decreasing function satisfying the following:
(1) $\varphi: [0, \infty) \to [0, \infty)$ is non decreasing function and $\varphi(t) > 0$ for $t > 0$
(2) $d(Tx, Ty) \leq d(x, y) - \varphi\big(d(x, y)\big)$ for all $x, y \in X$ whenever x, y are comparable
(3) there exists $x_0 \in X$ such that $x_0 \geq Tx_0$
(4) either T is continuous or X has the following property:

If $\{z_n\}$ is a decreasing sequence with $z_n \to z$, then $z_n \geq z\ \forall\ n \geq 1$

Then T has a fixed point.

Combing Theorems 2.5 and 2.11, we have the final version of our main theorem.

Theorem: 2.12 Let $(X, \leq)$ be a poset and (X, d) be a complete metric space. Suppose $T: X \to X$ is a non decreasing mapping satisfying the following:
(1) $\varphi: [0, \infty) \to [0, \infty)$ is non decreasing function and $\varphi(t) > 0$ for $t > 0$
(2) $d(Tx, Ty) \leq d(x, y) - \varphi\big(d(x, y)\big)$ for all $x, y \in X$ whenever x, y are comparable
(3) there exists $x_0 \in X$ such that x_0 and Tx_0 are comparable
(4) either T is continuous or X has the following properties:
 (i) if $\{z_n\}$ is an increasing sequence I X with $z_n \to z$, then $z_n \leq z\ \forall\ n \geq 1$ and
 (ii) if $\{z_n\}$ is a decreasing sequence in X with $z_n \to z$, then $z_n \geq z\ \forall\ n \geq 1$

Then T has a fixed point.

In fact, the sequence $\{x_n\}$ denoted by $x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots$ is a Cauchy sequence and hence converges, say to x, which is a fixed point of T.

Theorem: 2.13 Under the hypothesis of Theorem 2.5, suppose x is a fixed point of T. Suppose there exists $z \in X$ such that $z \leq x$. Then $x = \lim_{n\to\infty} z_n$ where $z_{n+1} = Tz_n, n = 0, 1, 2, \ldots$ where $z_0 = z$.

Proof: We have $z \leq x \Rightarrow Tz \leq Tx = x \Rightarrow z_1 \leq x$

By induction, we can show that $z_n \leq x\ \forall\ n = 0, 1, 2, \ldots$

$\therefore d(z_{n+1}, x) = d(Tz_n, Tx) \leq d(z_n, x) - \varphi\big(d(z_n, x)\big)$
$\leq d(z_n, x)$ …… (2.13.1) ($\because z_n$ and x are comparable)

There fore $\varphi(d(z_n, x)) \leq d(z_n, x) - d(z_{n+1}, x)$ (2.13.2)

By (2.13.1), $\{d(z_n, x)\}$ is a decreasing sequence and hence tends to a limit say δ.

So that $\delta \leq d(z_n, x)\ \forall\ n \geq 1$

Therefore $\varphi(\delta) \leq \varphi\big(d(z_n, x)\big)\ \forall\ n \geq 1$

$\therefore\ \varphi(\delta) \leq \varphi\big(d(z_n, x)\big) \leq d(z_n, x) - d(z_{n+1}, x)$ by (2.13.2)
$\to \delta - \delta = 0$ as $n \to \infty$
Hence $\varphi(\delta) = 0$

Therefore $\delta = 0$ (by (1) of hypothesis of Theorem 2.5)

$\therefore\ d(z_n, x) \to 0$ as $n \to \infty$
$\therefore\ z_n \to x$ as $n \to \infty$.

Theorem: 2.14 Under the hypothesis of Theorem 2.11, suppose x is a fixed point of T. Suppose there exists $z \in X$ such that $z \geq x$. Then $x = \lim_{n\to\infty} z_n$ where $z_{n+1} = Tz_n$, $n = 0, 1, 2, \ldots$ where $z_0 = z$.

Proof: We have $z \geq x \Rightarrow Tz \geq Tx = x \Rightarrow z_1 \geq x$

By induction, we can show that $z_n \geq x \;\forall\; n = 0, 1, 2, \ldots$

$$\therefore d(z_{n+1}, x) = d(Tz_n, Tx) \leq d(z_n, x) - \varphi\big(d(z_n, x)\big)$$
$$\leq d(z_n, x) \qquad (\because z_n \text{ and } x \text{ are comparable}) \tag{2.14.1}$$

There fore $\varphi(d(z_n, x)) \leq d(z_n, x) - d(z_{n+1}, x)$ (2.14.2)

By (2.14.1), $\{d(z_n, x)\}$ is a decreasing sequence and hence tends to a limit say δ.

So that $\delta \leq d(z_n, x) \;\forall\; n \geq 1$

Therefore $\varphi(\delta) \leq \varphi\big(d(z_n, x)\big) \;\forall\, n \geq 1$

$\therefore\; \varphi(\delta) \leq \varphi\big(d(z_n, x)\big) \leq d(z_n, x) - d(z_{n+1}, x)$ by (2.13.2) $\to \delta - \delta = 0$ as $n \to \infty$

Hence $\varphi(\delta) = 0$

Therefore $\delta = 0$ (by (1) of hypothesis of Theorem 2.11)

$\therefore\; d(z_n, x) \to 0$ as $n \to \infty$

$\therefore\; z_n \to x$ as $n \to \infty$.

Theorem: 2.15 Suppose the hypothesis of Theorem 2.12 holds. Further assume that, (H): if x and y are fixed points of T, then there exists $z \in X$ such that x, z are comparable and y, z are comparable. Then T has a unique fixed point.

Proof: By Theorem 2.12, T has a fixed point.

Suppose x, y are fixed points of T. By hypothesis, there exists $z \in X$ such that z is comparable with x and z is comparable with y.

Case (i): $z \leq x \text{ and } z \leq y$
Then by Theorem 2.13, we have $x = \lim_{n\to\infty} T^n z = y$. So that $x = y$.

Case: (ii) $z \leq x \text{ and } z \geq y$
Then by Theorem 2.13 and Theorem 2.14, we have $x = \lim_{n\to\infty} T^n z = y$. So that $x = y$.

Case: (iii) $z \geq x \text{ and } z \geq y$

Then by Theorem 2.14, we have $x = \lim_{n\to\infty} T^n z = y$. So that $x = y$.

Case: (iv) $z \geq x \text{ and } z \leq y$
Then by Theorem 2.13 and Theorem 2.14, we have $x = \lim_{n\to\infty} T^n z = y$. So that $x = y$.

Hence the theorem is proved.

Note: 2.16 If $(X, \leq)$ is a lattice, condition (H) is automatically satisfied.

The following example shows that the absence of (H) in Theorem 2.15 may result in more than one fixed point.

Example: 2.17 If $X = \{0,1\}$ with discrete ordering and discrete metric d. That is, 0 is not comparable with 1 and $d(0,1) = 1, d(1,1) = 0$
Define $T0 = 0$ and $T1 = 1$.
Take $\varphi(t) = \dfrac{t}{2}, \forall\, t \in [0, \infty)$.

Then all the conditions of Theorem 2.12 hold but (H) does not hold.

Further, T has two fixed points namely 0 and 1.

CONCLUSION

Under the hypothesis of Theorem 2.12 and in view of Theorem 2.15, the fixed point set $\mathfrak{F}$ of F decomposes the set X into pair wise disjoint sets $\{S_p \,/\, p \in \mathfrak{F}\}$ in the following way:

For $p \in \mathfrak{F}$, write $S_p = \{a \in X : p \text{ is comparable with } a\}$

Then (i) $S_p \neq \emptyset$, since $p \in S_p$

(ii) S_p and S_q are disjoint whenever $p,q \in \mathfrak{F}$ and $p \neq q$

(iii) $S = \bigcup\limits_{p \in \mathfrak{F}} S_p$ may be a proper sub set of X

in which case $X - S$ does not contain any fixed point of F.

This can be done by suitably adjusting the map T in Example 2.17.

REFERENCES

1. **Agarwal, R.P, El-gebeily. M.A. and O'regan. D:** Generalized contractions in partially ordered metric spaces, Applicable Analysis 87(1) (2008), 109-116.
2. **Amini_Harandi. A and Emami. H:** A fixed point theorem for contraction type maps in partially ordered metric spaces and applications to ordinary differential equations. Non-linear Analysis, 2010.
3. **Alber. Ya. I and Guerre-Delabriere. S:** New results in Operator Theory and its Applications: The Israel M. Glazman Memorial Volume, Birkhauser verlag. (1997).
4. **Aydi Hassen. Samet. B and Vetro Calogero:** Coupled fixed point results in cone-metric spaces for ω-compatible mappings, Fixed point Theory and Applications, 27, (2001), 1-15.
5. **Azam A and Shakeel. M:** Weakly contractive maps and common fixed points matematicki Vesnik, 60(2) (2008), 101-106.
6. Bae, J.S., Fixed point theorems for weakly contractive multivalued maps, J. Math. Anal. Appl., 284 (2003), 690-697.
7. **Gnana Bhaskar. T and Lakshmikantham. V:** Fixed point theorems in partially ordered metric spaces and applications. Non-linear Analysis, 2006, 65, 1379-1393.
8. **Choudhury. B.S and Amaresh Kundu. A:** coupled coincidence point result in partially ordered metric spaces for compatible mappings. Non-linear Analysis, 2010, 73, 2524-2531.
9. **Choudhury. B.S, Metiya. N and Amaresh Kundu:** Coupled coincidence point theorems in ordered metric spaces. Anna. Univ. Ferrara, 2011, 57, 1-16.
10. **Ciric Lj. B, Caki. N, Rajovic. M. and Ume. J.S:** Monotone generalized nonlinear contractions in partially ordered metric spaces, Fixed point theory and Applications, Article ID 131294,(2008), 1-11
11. **Doric. D:** Common fixed point for generalized (ψ,φ)-weak contraction. Applied Mathematics Letters, 2009, 22, 1896-1900.
12. **Harjani. J. and Sadarangani. K:** Fixed point theorems for weakly contractive mappings in partially ordered metric spaces, TMA, 71 (2009) 3403-3410.
13. **Jaggi. D.S. and Dass. B.K:** An extension of Banach's fixed point through rational expressions, Bull. Cal. Math. Soc., 72 (1980), 261-262.
14. **Kameswari. M.V.R:** Fixed point theorems of weakly contractive map via rational expressions in partially ordered metric spaces, International J. of Math. Sci. and Engg. Appls., 6 (2012), 1-19.
15. **Luoung. N.L. and Thuan N.X:** Fixed point theorem for generalized weak contractions satisfying rational expressions in ordered metric spaces, Fixed Point Theory and Applications (2001), 2011.46.
16. **Lakshmikantham. V and Ljubomir Ciric:** Coupled fixed point theorems for non-linear contractions in partially ordered metric spaces, Non-linear Analysis, 2009, 70, 4341-4349.
17. **Mehta and Joshi:** On coupled fixed point theorem in partially ordered complete metric spaces, Int. J. Pure Appl. Trchnol., 1(2) (2010), 87-92.
18. **Nashine. H.K, Samet B and Kim. J.K:** Fixed point results for contractions involving generalized altering distances in ordered metric spaces, Fixed Point Theory and Applications, 5 (2011), 1-16.
19. **Nieto. J.J and Rodriguez Lopez. R:** Contractive mapping theorems in partially ordered spaces and applications to ordinary differential equations. Order. 2005, 22(3), 223-239.

20. **Ran. A.C.M and Reurings. B.C.B.:** A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Amer. Math. Soc., 2003,132, 1435-144.
21. **Regan. D.O and Petrusel A:** Fixed point theorems for generalized contractions in ordered metric spaces, J. Math. Anal. Appl., 341 (2008), 1241-1252.
22. **Rhoades. B.E:** Some theorems on weakly contractive maps, Nonlinear Anal., 47 (2001), 2683-2693.
23. **Samet. B:** Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal., TMA, 72 (2009), 4508-4517.

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