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SUBORBITAL GRAPHS AND THEIR PROPERTIES FOR UNORDERED PAIRS IN $A_n(n = 5, 6, 7, 8)$ THROUGH RANK AND SUBDEGREE DETERMINATION

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ABSTRACT

In this paper, through computation of the rank and subdegrees of alternating group A_n $(n = 5, 6, 7, 8)$ on unordered pairs we construct the suborbital graphs corresponding to the suborbits of these pairs. When $A_n(n \geq 5)$ acts on unordered pairs the suborbital graphs Γ_1 and Γ_2 corresponding to the non-trivial suborbits Δ_1 and Δ_2 are found to be connected, regular and have undirected edge. Further, we investigate properties of the suborbital graphs constructed.

Mathematics Subject Classification: *06A75; 20G07; 20B05.*

Key Words: *Rank, subdegrees, unordered pair of an alternating group, suborbital graphs.*

1. PRELIMINARIES

1.1 Basic notions and terminology

We first present some basic notions and terminologies as used in the context of graphs and suborbital graphs that shall be used in the sequel

A_n -Alternating group of degree n and order $\frac{n!}{2}$;
$|G|$ -The order of a group G;
$X^{(2)}$ -The set of an unordered pairs from set $X = \{1, 2, \ldots, n\}$;
$\{t, q\}$ -Unordered pair;

Definition: 1.1.1 A graph is a diagram consisting of a set V whose elements are called vertices, nodes or points and a set E of unordered pairs of vertices called edges or lines. We denote such a graph by $G(V, E)$ or simply by G if there is no ambiguity of V and E.

Definition: 1.1.2 Two vertices u and v of a graph $G(V, E)$ are said to be adjacent if there is an edge joining them. This is denoted by $\{u, v\}$ and sometimes by uv. In this case u and v are said to be incident to such edge.

Definition: 1.1.3 A graph consisting of one vertex and no edge is called a trivial graph.

Definition: 1.1.4 A graph whose edge set is empty is called a null graph.

Definition: 1.1.5 The degree (valency) of a vertex v of $G(V, E)$ is the number of edges incident to v.

Definition: 1.1.6 A graph $G(V, E)$ is said to be connected if there is a path between any two of its vertices.

Definition: 1.1.7 The girth of a graph $G(V, E)$ is the length of the shortest cycle if any in $G(V, E)$.

Definition: 1.1.8 A graph in which every vertex has the same degree is called a regular graph.

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Definition: 1.1.9 Let G be transitive on X and let G_x be the stabilizer of a point $x \in X$. The orbits $\Delta_0 = \{x\}$, $\Delta_1, \Delta_2, \dots, \Delta_{r-1}$ of G_x on X are called the suborbits of G. The rank of G is r and the sizes

$\mathrm{n}_i = |\Delta_1| \; (i = 0,1,2,\dots,r-1)$ often called the lengths of the suborbits, are known as subdegrees of G.

It is worth noting that both r and the cardinalities of the suborbits $\Delta_i (i = 0, 1, 2, \dots, r-1)$ are independent of the choice of $x \in X$.

Definition: 1.1.10 Let Δ be an orbit of G_x on X. Define $\Delta^* = \{gx \mid g \in G, x \in g\Delta\}$, then Δ^* is also an orbit of G_x and is called the G_x-orbit (or the $G - suborbit$) paired with Δ. Clearly $|\Delta| = |\Delta^*|$

If $\Delta^* = \Delta$, then Δ is called a self-paired orbit of G_x.

Theorem: 1.1.11(Sims 1967, [6]) Let Γ_i^* be the suborbital graph corresponding to the suborbital O_i^*. Let the suborbits $\Delta_i \; (i = 0, 1, \dots, r-1)$ correspond to the suborbital O_i. Then Γ_i is undirected if Δ_i is self-paired and is directed if Δ_i is not self-paired.

1.2 INTRODUCTION

In 1967, Sims[6] introduced suborbital graphs corresponding to the non-trivial suborbits of a group. He called them orbitals. In1977, Neumann [4] extended the work of Higman [2] and Sims [6] to finite permutation groups, edge coloured graphs and Matrices. He constructed the famous Peterson graph as a suborbital graph corresponding to one of the nontrivial suborbits of S_5 acting on unordered pairs from the set $X = \{1, 2, 3, 4, 5\}$. The Peterson graph was first introduced by Petersen in 1898 [5].

In199, Kamuti[3] devised a method for constructing some of the suborbital graphs of $PSL\ (2, q)$ and $PGL\ (2, q)$ acting on the cosets of their Maximal dihedral sub-groups of orders $q - 1$ and $2(q - 1)$ respectively. This method gave an alternative way of constructing the Coxeter graph which was first constructed by Coxeter in 1986[1]. In this paper, through computation of the rank and subdegrees of alternating group A_n $(n = 5, 6, 7 ,8)$ on unordered pairs we construct the suborbital graphs corresponding to the suborbits of these pairs and further investigate properties of the suborbital graphs constructed.

2. SUBORBITAL GRAPHS OF $G = A_n$ ACTING ON $\mathrm{X}^{(2)}$

In this section we construct and discuss the properties of the suborbital graphs of $G = A_n$ acting on $\mathrm{X}^{(2)}$.

2.1 The suborbital graphs of $G = A_5$ acting on $X^{(2)}$

The three orbits of $G_{\{1,2\}}$ acting on $\mathrm{X}^{(2))}$ are;

$\mathrm{Orb}_{\mathrm{G}_{\{1,2\}}}\{1,2\} = \{\{1,2\}\} = \Delta_0$, the trivial orbit.

$\mathrm{Orb}_{\mathrm{G}_{\{1,2\}}}\{1,3\} = \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}\} = \Delta_1$, the set of all unordered pairs containing exactly one of 1 and 2.

$\mathrm{Orb}_{\mathrm{G}_{\{1,2\}}}\{3,4\} = \big\{\{3,4\}, \{3,5\}, \{4,5\}\big\} = \Delta_2$, the set of all unordered pairs containing neither 1 nor 2.

The suborbital graph corresponding to Δ_0 is the null graph since its edge set is empty.

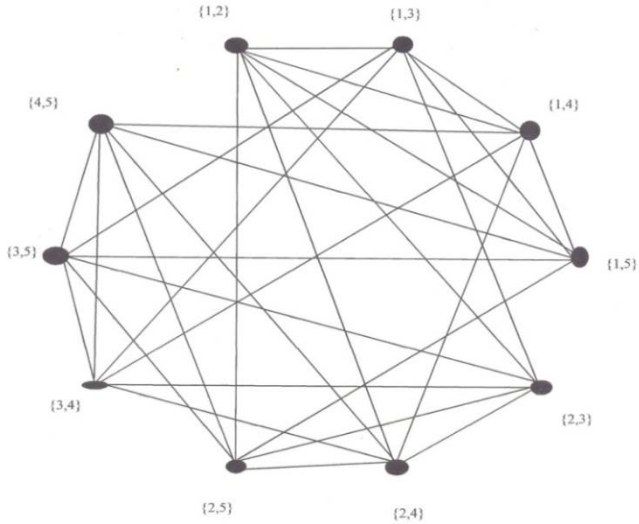
We now consider the non-trivial suborbit Δ_1 and Δ_2. By Definition 1.1.10, Δ_1 and Δ_2 are self-paired and hence by Theorem 1.1.11 their corresponding suborbital graphs Γ_1 and Γ_2 are undirected.

The suborbital O_1 corresponding to the suborbit Δ_1 is, $O_1 = \{(g\{1,2\}, g\{1,3\}) \,\big|\, g \epsilon G\}$. Thus the corresponding suborbital graph Γ_1 has two $2 - elements$ subsets S and T from $X = \{1, 2, 3, 4, 5\}$ adjacent if and only if $|S \cap T| = 1$.

Similarly the suborbital O_2 corresponding to the suborbit Δ_2 is $O_2 = \{(g\{1,2\}, g\{3,4\}) \mid g \in G]$. Thus the corresponding suborbital graph Γ_2 has two $2 - elements$ subset S and T from

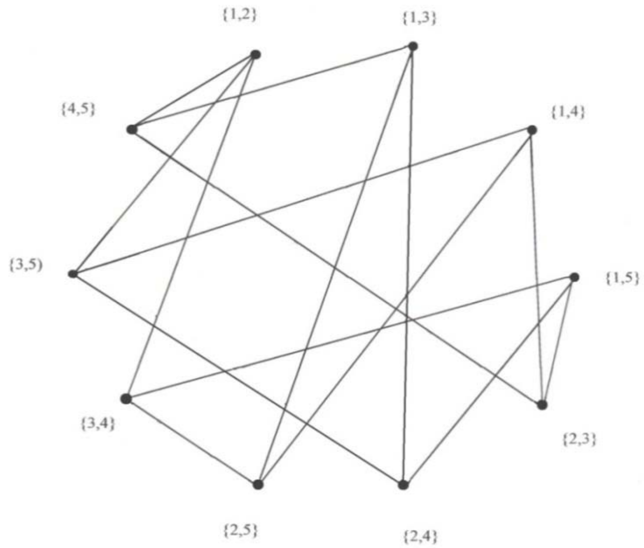
$X = \{1, 2, 3, 4, 5\}$ Adjacent if and only if $|S \cap T| = 0$.

Figure 2.1.1: The suborbital graph $\Gamma_1$corresponding to the suborbitΔ_1 of $G = A_5$ on $X^{(2)}$



From the diagram we see that$\Gamma_1$is regular of degree 6, it is a connected graph and has girth 3 since $\{1,2\}, \{1,3\}$ and $\{2,3\}$ are joined by a closed path.

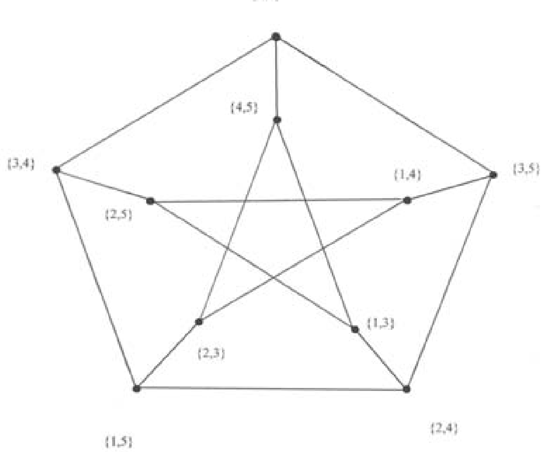
Figure 2.1.2 (a): The suborbital graph Γ_2 corresponding to the suborbit Δ_2 of $G = A_5$ on $X^{(2)}$



We see that Γ_1 is regular of degree 3. It is a connected graph of girth 5 since there exist a path joining vertices {1,2}, {3,5}, {2,4}, {1,5} and {3,4}.

It can also be represented as shown in the figure below;

Figure 2.1.2 (b): The suborbital graph Γ_2 corresponding to the suborbit Δ_2 of $G = A_5$



The suborbital graphΓ_2 is the famous Petersen graph and is regular of degree 3. It is a connected graph and its girth is 5 since there exists a cycle through$\{1,2\}, \{3.5\}, \{2,4\}, \{1,5\}$ and $\{3,4\}$.

2.2 The suborbital graphs of $G = \mathrm{A}_6$ acting on $X^{(2)}$

The orbits of $G_{\{1,2\}}$ acting on $X^{(2)}$ are;

$\mathrm{Orb}_{G_{\{1,2\}}}\{1,2\} = \{\{1,2\}\} = \Delta_0$, the trivial orbit.

$\mathrm{Orb}_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}\} = \Delta_1$, the set of all unordered pairs containing exactly one of 1 and 2.

$\mathrm{Orb}_{G_{\{1,2\}}}\{3,4\} = \{\{3,4\}, \{3,5\}, \{3,6\}, \{4,5\}, \{4,6\}, \{5,6\}\} = \Delta_2$, the set of all unordered pairs containing neither 1 nor 2.

We now discuss the suborbital graphs corresponding to these suborbits. The suborbital graph corresponding to Δ_0 is the null graph.

Next we consider the non-trivial suborbitsΔ_1 andΔ_2. By Definition 1.1.10, Δ_1 and Δ_2 are self-paired and hence by Theorem 1.1.11 their corresponding suborbital graphs Γ_1, and Γ_2,are undirected.

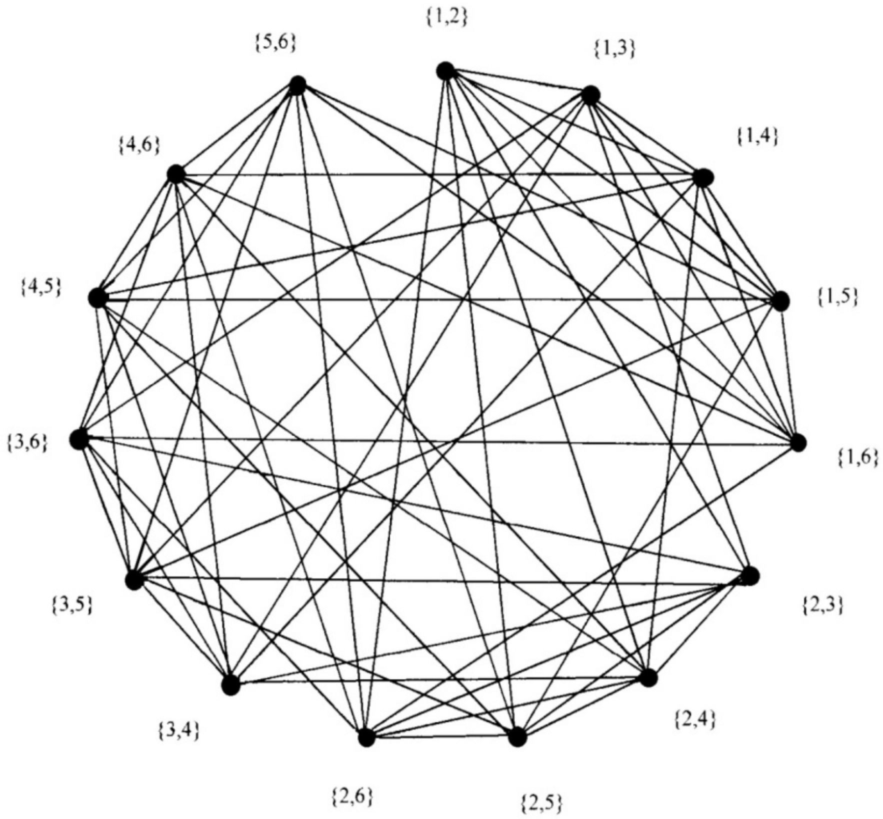
The suborbitalO_1 corresponding to the suborbit $\Delta_1$is $\mathrm{O}_1 = \{(g\{1,2\}, g\{1,3\}) \mid g\epsilon G\}$. Thus thecorresponding suborbital graph Γ_1 has two $2-elements$ subsets S and T from $X = \{1, 2, 3, 4, 5, 6\}$ adjacent if and only if $|S \cap T| = 1$.

Similarly the suborbital O_2 corresponding to the suborbit Δ_2 is $\mathrm{O}_2 = \{(g\{1,2\}, g\{3,4\}) \mid g\epsilon G\}$. Thus the corresponding suborbital graph Γ_2 has two $2-\ elements$ subsets $S\ and\ T$ from

$X = \{1, 2, 3, 4, 5, 6\}$ adjacent if and only if $|S \cap T| = 0$.

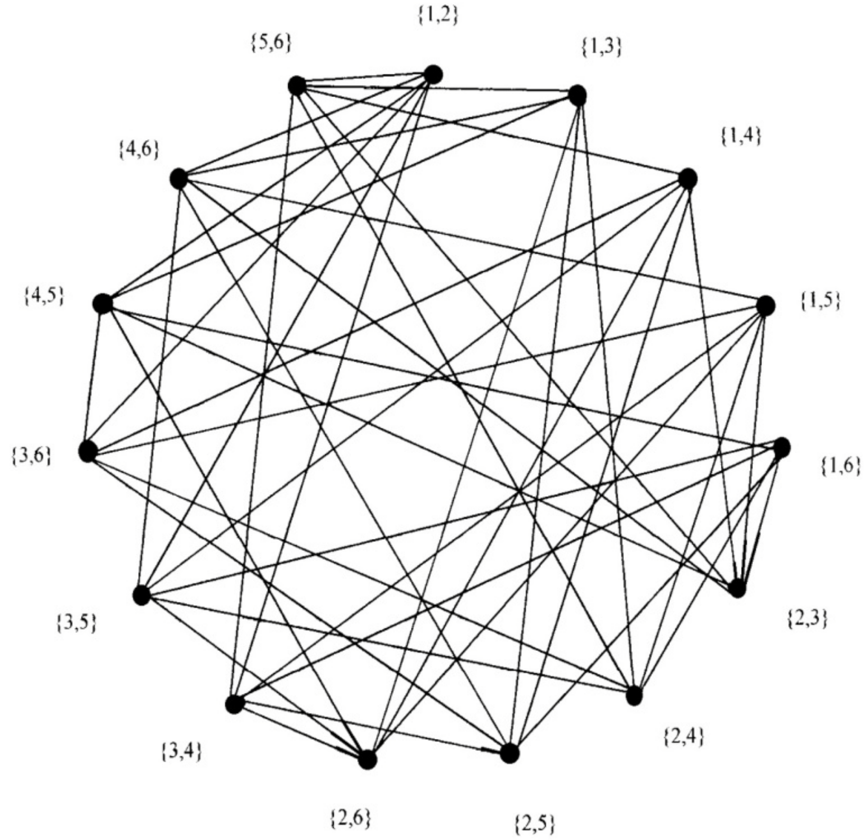
The properties of the suborbital graph $\Gamma_1$and $\Gamma_2$can be investigated by constructing these as follows;

Figure 2.2.1: The suborbital graph $\Gamma_1$corresponding to the suborbit Δ_1 of $G = \mathrm{A}_6$ on $X^{(2)}$



Clearly$\Gamma_1$is regular of degree 8 and is a connected graph. Its girth is 3 since $\{1,2\}, \{1,4\}$, and$\{2,4\}$ are joined by a closed path.

Figure 2.2.2: The suborbital graph Γ_2 corresponding to the suborbit Δ_2 of $G = A_6$on $X^{(2)}$



From the diagram we see that $\Gamma_2$is regular of degree 6 and is connected. Its girth is 3 since $\{1,2\}, \{5,6\}$ and $\{3,4\}$ are joined by a closed path.

2.3 The suborbital graphs of G = A_7 acting $X^{(2)}$

The orbits of $G_{\{1,2\}}$ acting on $X^{(2)}$ are;

$\mathrm{Orb}_{G_{\{1,2\}}}\{1,2\} = \{1,2\} = \Delta_0$, the trivial orbit.

$\mathrm{Orb}_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{1,7\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{2,7\}\} = \Delta_1$, the set of all unordered pairs containing exactly one of 1 and 2.

$\mathrm{Orb}_{G_{\{1,2\}}}\{3,4\} = \{\{3,4\}, \{3,5\}, \{3,6\}, \{3,7\}, \{4,5\}, \{4,6\}, \{4,7\}, \{5,6\}, \{5,7\}, \{6,7\}\} = \Delta_2$, the set of all unordered pairs containing neither 1 nor 2.

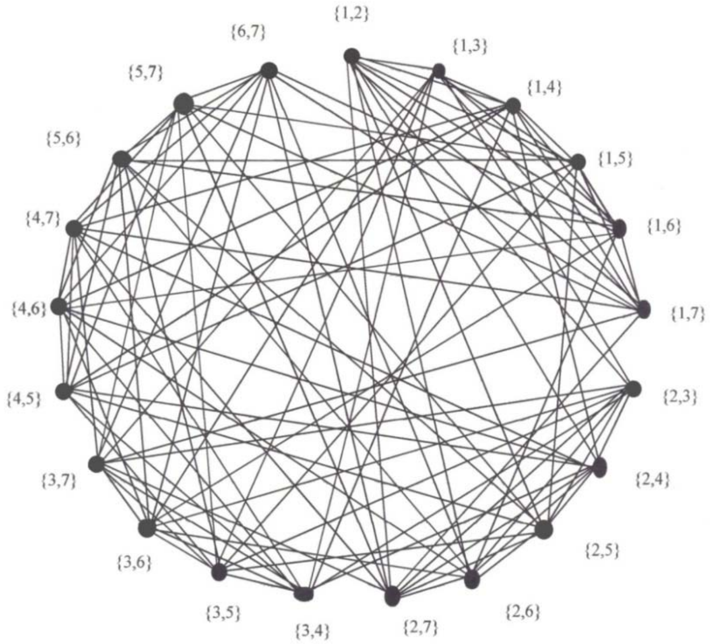
Next we the discuss suborbital graphs corresponding to these suborbits. The suborbital graph corresponding to Δ_0 is the null graph.

We now consider the non-trivial suborbits Δ_1 and Δ_2. By Definition 1.1.10 Δ_1 and Δ_2 are self-paired and hence by Theorem 1.1.11 their corresponding suborbital graphs $\Gamma_1$and Γ_2 are undirected.

The suborbital O_1 corresponding to the suborbit $\Delta_1$is $O_1 = \{(g\{1,2\}, g\{1,3\}) \mid g \epsilon G\}$. Thus the corresponding suborbital graphΓ_1 has two $2 - elements$ subsets S and T from $X = \{1, 2, 3, 4, 5, 6, 7\}$ adjacent if and only if $|S \cap T| = 1$.

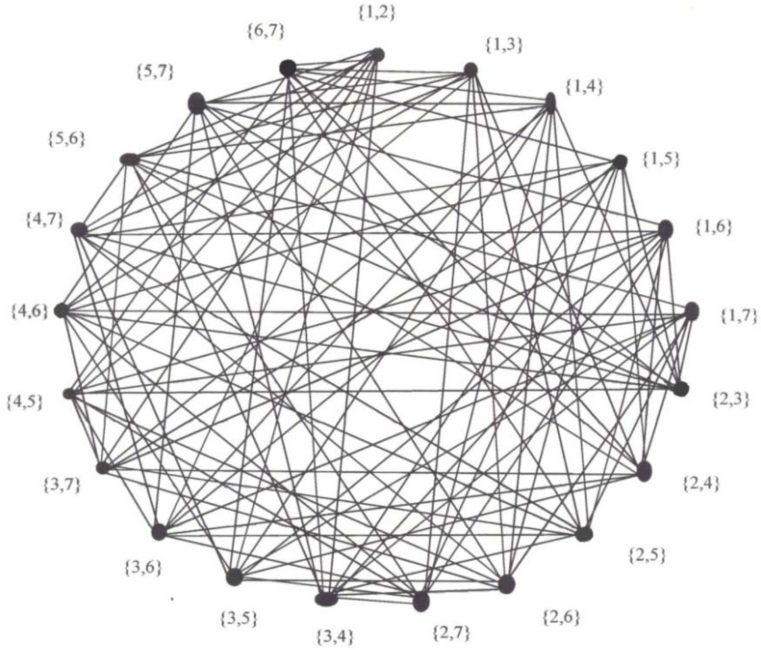
Similarly the suborbital O_2 corresponding to the suborbit Δ_2 is $O_2 = \{(g\{1,2\}, g\{3,4\}) | g \epsilon G\}$. Therefore the corresponding suborbital graph Γ_2 has two $2 - elements$ subsets S and T from $X = \{1, 2, 3, 4, 5, 6, 7\}$ adjacent if and only if $|S \cap T| = 0$.The properties of the suborbital graph $\Gamma_1$and $\Gamma_2$can be studied by constructing them as follows;

Figure 2.3.1: The suborbital graph $\Gamma_1$corresponding to the suborbit Δ_1 of $G = A_7$ on$X^{(2)}$



From the figure above we see that Γ_1 is regular of degree 10 . It is also connected and has girth 3 since $\{1,2\}, \{1,3\}$ and $\{2,3\}$ are joined by a closed path.

Figure 2.3.2: The suborbital graph $\Gamma_2$corresponding to the suborbit Δ_2 of $G = A_7$ on $X^{(2)}$



From the figure above we see that $\Gamma_2$is regular of degree 10. It is a connected graph of girth 3 since there exist a cycle through vertices $\{1,2\}, \{6,7\}$ and $\{4,5\}$.

2.4 The suborbital graph of $G = A_8$ acting on $X^{(2)}$

The three orbits of $G_{\{1,2\}}$ acting on $X^{(2)}$ are;

$\mathrm{Orb}_{G_{\{1,2\}}}\{1,2\} = \{\{1,2\}\} = \Delta_0$, the trivial orbit.

$\mathrm{Orb}_{G_{\{1,2\}}}\{1,3\} = \{\{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{1,7\}, \{1,8\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{2,7\},$

$\{2,8\} = \Delta_1$, the set of all unordered pairs containing exactly one of 1 and 2.$= \Delta_1$
$\mathrm{Orb}_{G_{\{1,2\}}}\{3,4\} = \{\{3,4\}, \{3,5\}, \{3,6\}, \{3,7\}, \{3,8\},\{4,5\}, \{4,6\}, \{4,7\}, \{4,8\}, \{5,6\}, \{5,7\}, \{5,8\}, \{6,7\}, \{6,8\}, \{7,8\} = \Delta_2$.,
the set of all unordered pairs containing neither 1 nor 2.

Next we discuss the suborbits Δ_0, Δ_1 and Δ_2 and their corresponding suborbital graphs.

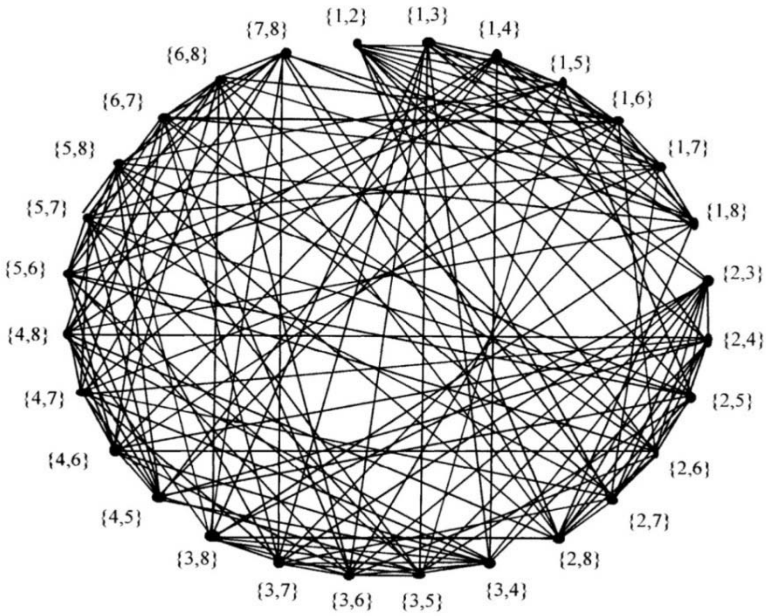
The suborbital graph corresponding to Δ_0 is the null graph. We next consider the suborbits Δ_1 andΔ_2, the non-trivial orbits.

By Definition 1.1.10, Δ_1 and Δ_2 are self-paired and hence by Theorem 1.1.11 their corresponding suborbital graphs $\Gamma_1$and $\Gamma_2$are undirected. The suborbitalO_1corresponding to the suborbit Δ_1 is $O_1 = \{(g\{1,2\}, g\{1,3\}) \mid g \epsilon G\}$. Thus the corresponding suborbital graph Γ_1 has two $2 - elements$ subsets S and T from $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ adjacent if and only if $|S \cap T| = 1$.

Similarly the suborbital O_2 corresponding to the suborbit Δ_2 is $O_2 = \{(g\{1,2\}, g\{3,4\}) \mid g \epsilon G\}$. Therefore the corresponding suborbital graph Γ_2 has two elements subsets S and T from $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ adjacent if and only if $|S \cap T| = 0$.

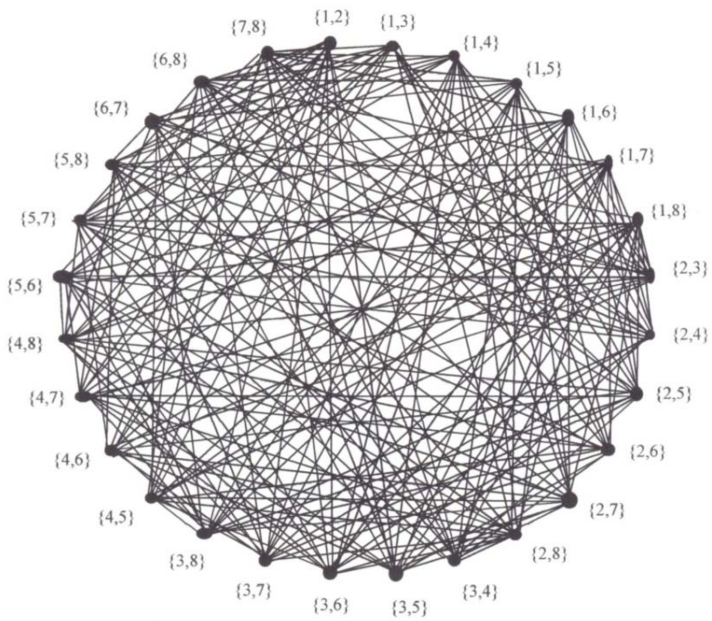
The properties of the suborbital graph $\Gamma_1$and $\Gamma_2$can be studied by constructing themas follows;

Figure 2.4.1: The suborbital graph $\Gamma_1$corresponding to the suborbit Δ_1 of $G = A_8$ on $X^{(2)}$



From the diagram, we see that Γ_1 is regular of degree 12 . It is a connected graph and has girth 3 since $\{3,4\}, \{3,5\}$ *and* $\{4,5\}$ are joined by a closed path.

Figure 2.4.2: Suborbital graph $\Gamma_2$corresponding to the suborbit Δ_2 of $G = A_8$ on $X^{(2)}$



Clearly $\Gamma_2$is regular of degree 15 and is a connected graph and its girth is 3 since $\{1,2\}, \{7,8\}$ and $\{3,4\}$are joined by closed path.

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