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COMMON FIXED POINT THEOREM OF COMPATIBLE OF TYPE (P) USING IMPLICIT RELATION IN FUZZY METRIC SPACE

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ABSTRACT

In this paper we prove a common fixed print theorem for compatible mapping of type (P) in Fuzzy metric space using implicit relation. Our result modifies the results of M. Koireng et.al. [10].

Mathematical Classification*: 54H25, 54E50.*

Keywords*: Compatible Maps, Fuzzy Metric Spaces, Compatible Maps of Type (P), Implicit Relation.*

INTRODUCTION

The concept of fuzzy sets was introduced initially by Zadeh [17] which laid the foundation of fuzzy mathematics. George and Veeramani In [5] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [9]. They also obtained that every metric space induces a fuzzy metric spaces. Sessa [16] proved a generalization of commutativity. So called weak commutatively. Futher Jungek [8] more generalized commutativity called compatibility in metric space.

In [1] Cho, Sharma et al introduced the concept of semi compatibility in D-metric space. Recently Bijendra Singh *et al* [15] introduced the concept of semi compatible mapping in the context of a fuzzy metric space.

The first important result of compatible mapping was obtained by jungck [8].pathak, chang and cho introduced the concept of compatible mapping of type (P) [12]

Our aim in this paper is to prove some common fixed point theorem of compatible map of type (P) by generalized some interesting result [2] [10].

2. PRETIMINARIES AND DEFINATION

Definition 2.1 [6]: A binary operation *: [0, 1] × [0, 1] ⟶ [0, 1] is called a continuous t-norm if ([0, 1], *) is an abelian topological monoid with 1 such that $a*b \le c*d$. Whenever $a \le c$, $b \le d$ for all $a,b,c,d \in [0,1]$ examples of t-norm are $a*b = ab$ and $a*b$ = $\min\{a,b\}$

Definition 2.2 [5]: the 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$. Satisfying the following conditions:

(1) $M(x,y,t) > 0$

(2) $M(x,y,t) = 1$ If and only if $x = y$

(3) $M(x,y,t) = M(y,x,t)$

(4) $M(x,y,t)*M(y,z,s) \le M(x,z,t+s)$

(5) $M(x,y,.):(0,\infty) \to [0,1]$ Is continuous, for all $x,y,z \in X$ and $t,s > 0$

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Let (X,d) be a metric space, and let $a*b = ab$ or $a*b = \min\{a,b\}$. Let $M(x,y,t) = \dfrac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space.

Definition 2.3 [14]: A sequence $\{x_n\}$ in a fuzzy metric space $\{X, M, *)$ is said to be a Cauchy sequence if and only if for each $\in > 0$, $t > 0$, there exists $x \in N$ such that $M(x_n, x_m, t) > 1- \in$ For all $n, m \geq x_0$

The sequence $\{x_n\}$ is said to converge to a point x in X iff for each $\in > 0$, $t > 0$ there exists $x_0 \in N$ such that $M(x_n, x, t) > 1- \in$ For all $n \geq x_0$

A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence in it converges to a point in it.

Definition 2.4 [15]: A pair of self mappings (A, S) of fuzzy metric space $(X, M, *)$ is said to be compatible if

$$\underset{n\to\infty}{Lim} M(ASx_n, SAx_n, t) \to 1 \forall t > 0$$

Whenever $\{x_n\}$ is a sequence in X such that $\underset{n\to\infty}{Lim} Sx_n = \underset{n\to\infty}{Lim} Ax_n = x$, for some $x \in X$

Definition 2.5 [14]: A pair (A, S) of self mappings of a fuzzy metric space is said to be semi compatible if $\underset{n\to\infty}{Lim} ASx_n = Sx$ whenever $\{x_n\}$ is a sequence in X such that $\underset{n\to\infty}{Lim} ASx_n = \underset{n\to\infty}{Lim} Sx_n = x$ so (A, S) is semi compatible and $Ay = Sy \Rightarrow ASy = SAy$ by taking $\{x_n\} = y$ and $x = Ay = Sy$.

Proposition 2.1 [2]: in a fuzzy metric space $(X, M, *)$ limit of a sequence is unique.

Proof: Let $\{x_n\} \to x$ and $\{x_n\} \to y$ then $\underset{n\to\infty}{Lim} M(x_n, x, t) = 1 = \underset{n\to\infty}{Lim} M(x_n, y, t)$

Now $M(x, y, t) \geq M(x, x_n, t/2) * M(y, x_n, t/2)$ taking Limit $n \to \infty$, $M(x, y, t) \geq 1*1$

i.e. $M(x, y, t) = 1$ for all $t > 0$ thus $x = y$ and hence the limit is unique

Proposition 2.2 [15]: (A, S) is a semi-compatible pair of self maps of a fuzzy metric space $(X, M, *)$ and S in continuous then (A, S) is compatible.

Proof: Consider a sequence $\{x_n\}$ in X such that $\{Ax_n\} \to x$ and $\{Sx_n\} \to x$, by semi compatibility of (A, S) we have $\underset{n\to\infty}{Lim} ASx_n = Sx$. As S is continuous we get $\underset{n\to\infty}{Lim} SAx_n = Sx$

Now, $\underset{n\to\infty}{Lim}(SAx_n, ASx_n, t) = M(Sx, Sx, t) = 1$

Hence (A, S) is compatible.

Note: Converse is not true.

Definition 2.6 [12]: Self mappings *A* and *S* of a fuzzy metric space (*X, M*, ∗) is said to be compatible of type (P) if $Lim \ \{x_n\} \to y$ then $\underset{n\to\infty}{Lim} M(AAx_n, SSx, t) = 1$ For all t > 0

Whenever { x_n } is a sequence in *X* such that $\underset{n\to\infty}{Lim} Ax_n = \underset{n\to\infty}{Lim} Sx_n = z$ For some z ∈ *X*.

Lemma [15]: let (X, M, *) be a fuzzy metric space. If there exists $k \in X$ such that M(x, y, kt) ≥M(x, y, t/k n) for positive integer n .taking limit as n → ∞, M(x, y, kt) ≥1and hence x=y

Lemma 2.8 [14]: the only t-norm * satisfying r*r ≥ r for all r∈[0,1] is the minimum t-norm, that is, a* b = min {a, b} for all a,b ∈[0,1]

Proposition 2.10 [11]: Let (X, M, *) be a fuzzy metric space and let A and S be Continuous mappings of X then A and S are compatible if and only if they are Compatible of type (P).

Proposition 2.11 [12]: Let (X, M, *) be a fuzzy metric space and let A and S be Compatible mappings of type (P) and Az=Sz for some z∈ X, then AAz=ASz=SAz=SSz.

Proposition 2.12 [10]: Let (X, M, *) be a fuzzy metric space and let A and S be Compatible mappings of type (P) and let A x_n , S x_n → z as n →∞ for some z∈ X. Then

(i) $\lim_{n\to\infty} SSx_n = Az$ For if A is continuous at z,

(ii) $\lim_{n\to\infty} AAx_n = Sz$ For if S is continuous at z,

(iii) ASz=SAz and Az=Sz if A and S are continuous at z.

A CLASS OF IMPLICIT RELATION

Let ϕ be the set of all real and continuous from, $\varphi : [0,1]^s \to R$ satisfying the following conditions.

$(A-1)$ φ is non-increasing in second, third, fourth and fifth argument

$(A-2)$ $\varphi(u,v,v,u,v) \geq 0 \Rightarrow u \geq v$

$\varphi(u,v,v,v,v) \geq 0 \Rightarrow u \geq v$

Example: $\varphi(t_1,t_2,t_3,t_4,t_5) = t_1 - Max.\{t_2,t_3,t_4,t_5\}$

3. MAIN RESULT

Theorem 3.1: Let A, B, S and T be self mappings of a complete fuzzy metric space $(X, M, *)$ with continuous t-norm defined by $a*b = \min\{a*b\}\{b\} \in [0,1]$ Satisfying

(i) $A(X) \subset T(X)$, $B(X) \subset S(X)$

(ii) S and T are continuous.

(iii) Pairs (A, S) and (B, T) are compatible of type (P)

(iv) ∃ Some $k \in (0,1)$ such that for all $x, y \in X, t > 0$

$$\varphi\big(M(Ax,By,kt), M(Sx,Ty,t), M(Sx,Ax,t), M(Ty,By,kt), M(Ty,Ax,t)\big) \geq 0$$

(v) $\forall x, y \in X, M(x,y,t) \to 1$ As $t \to \infty$

Then A, B, S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be any point as $A(X) \subset T(X)$ and $S(X) \subset B(X)$, $\exists x_1 \in X$ and $x_2 \in X$ such hat $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively we construct a sequence $\{y_n\}$ in X such that

$y_{2n+1} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$; $(y_{2n} = Sx_2 n)$ n = 0 1, With $x = x_{2n}, y = x_{2n+1}$ using contractive condition, we get

$$\varphi(M(Ax_{2n},Bx_{2n+1},kt), M(Sx_{2n},Tx_{2n+1},t), M(Sx_{2n},Ax_{2n},t), M(Tx_{2n+1},Bx_{2n+1},kt), M(Tx_{2n+1},Ax_{2n},t)) \geq 0$$

$$\Rightarrow \varphi\,(M(y_{2n+1},y_{2n+2},kt), M(y_{2n},y_{2n+1},t), M(y_{2n},y_{2n+1},t), M(y_{2n+1},y_{2n+2},kt), M(y_{2n+1},y_{2n+1},t)) \geq 0$$

$$\Rightarrow \ \varphi \ (M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, kt), 1) \geq 0$$

Since ϕ is non-increasing in fifth argument therefore,

$$\varphi \ (M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n+1}, y_{2n+1}, t)) \geq 0$$

Therefore by (2) property of implicit relation

$$M(y_{2n+1}, y_{2n}, kt) \geq M(y_{2n+1}, y_{2n}, t)$$

Similarly $M(y_{2n+1}, y_{2n}, kt) \geq M(y_{2n}, y_{2n-1}, t)$

Hence $M(y_{n+1}, y_n, kt) \geq M(y_n, y_{n-1}, t) \forall n$

We show that

$\underset{n\to\infty}{Lim} M(y_{n+p}, y_n, t) = 1$ For all p and t > 0

Now

$$\begin{aligned} M(y_{n+1}, y_n, t) \ &\geq M(y_n, y_{n-1}, t/k) \\ &\geq M(y_n, y_{n-2}, t/k^2) \\ &\geq ..... \\ &\geq M(y_1, y_0, t/k^n) \to 1, \text{ As } t/k^2 \to \infty \text{ as } n \to \infty \end{aligned}$$

Thus the result holds for $p = 1$. By induction hypothesis suppose that the result hold for p = r, now.

$$M(y_n, y_{n+r+1}, t) \geq M(y_n, y_{n+r}, t/2) * M(y_{n+r}, y_{n+r+1}, t/2) \to 1*1 = 1$$

Thus the result holds for $p = r + 1$

Hence $\{y_n\}$ is a Cauchy sequence in X and as X is complete we get $\{y_n\} \to z \in X$. Hence

$$Ax_{2n} \to z, Sx_{2n} \to z \quad ... \text{ (I)}$$
$$Tx_{2n+1} \to z, Bx_{2n+1} \to z \text{ .... (II)}$$

From proposition and since pairs (A, S) and (B, T) are compatible of type (P) we get

AA $x_{2n} \to Sz$, $SSx_{2n} \to Az$, $BBx_{2n+1} \to Tz$, $TTx_{2n+1} \to Bz$

From contractive condition we get

$$\begin{aligned} \varphi(M(AAx_{2n}, BBx_{2n+1}, kt), M(SAx_{2n}, TBx_{2n+1}, t), \ & M(AAx_{2n}, SAx_{2n}, t), M(BBx_{2n+1}, TBx_{2n+1}, kt) \\ & M(AAx_{2n+1}, TBx_{2n}, t)) \geq 0 \end{aligned}$$

Taking limit as $n \to \infty$ we get

$$\varphi(M(Sz, Tz, kt), M(Sz, Tz, t), M(Sz, Sz, t), M(Tz, Tz, kt), M(Sz, Tz, t)) \geq 0$$

$$\Rightarrow \ \varphi(M(Sz, Tz, kt), M(Sz, Tz, t), 1, 1, M(Sz, Tz, t)) \geq 0$$

Since φ is non increasing in third, fourth argument

$$\Rightarrow \phi(M(Sz,Tz,kt), M(Sz,Tz,t), M(Sz,Tz,t), M(Sz,Tz,kt), M(Sz,Tz,t)) \geq 0$$

$$\Rightarrow M(Sz,Tz,kt) \geq M(Sz,Tz,t)$$

$\Rightarrow$ Sz =Tz (by Lemma)

By From contractive condition

$$\phi(M(Az,BTx_{2n+1},kt), M(Sz,TTx_{2n+1},t), M(Az,Sz,t), M(BTx_{2n+1},TTx_{2n+1},t), M(TTx_{2n+1},Az,t) \geq 0 \text{ as } n \to \infty$$

$$\phi(M(Az,Tz,kt), M(Sz,Sz,t), M(Az,Tz,t), M(Tz,Tz,kt), M(Az,Tz,t)) \geq 0$$

$$\phi(M(Az,Tz,kt),1, M(Az,Tz,t),1, M(Az,Tz,t)) \geq 0$$

$\Rightarrow$ Since ϕ is non-increasing in second and fourth argument

$$\phi(M(Az,Tz,kt), M(Az,Tz,t), M(Az,Tz,t), M(Az,Tz,t), M(Az,Tz,t) \geq 0$$

$$\Rightarrow M(Az,Tz,kt) \geq M(Az,Tz,t)$$

$\Rightarrow$ $Az = Tz = Sz$ [By Lemma]

Again from contractive condition

$$\phi(M(Az,Bz,kt), M(Sz,Tu,t), M(Az,Sz,t), M(Tz,Bz,kt), M(Tz,Az,t) \geq 0$$

$$\Rightarrow \phi(M(Az,Bz,kt), M(Az,Az,t), M(Az,Az,t), M(Az,Bz,kt), M(Az,Az,t) \geq 0$$

$$\Rightarrow \phi(M(Az,Bz,kt),1,1, M(Az,Bz,kt),1) \geq 0$$

Since ϕ is non increasing in second, third and fifth argument

$$\Rightarrow \phi(M(Az,Bz,kt), M(Az,Bz,t), M(Az,Bz,t), M(Az,Bz,kt), M(Az,Bz,t) \geq 0$$

$$\Rightarrow \phi(M(Az,Bz,kt) \geq M(Az,Bz,t)$$

$$\Rightarrow Az = Bz$$

And so $\Rightarrow$ $Az = Tz = Sz$ =BZ and now we show that BZ=z

By contractive condition

$$\phi(M(z,Bz,kt), M(z,Tz,t), M(z,z,t), \; M(Tz,Bz,kt), M(z,Tz,t)) \geq 0$$

$$\Rightarrow \phi(M(z,Bz,kt), M(z,Bz,t),1,1, M(Bz,z,t) \geq 0$$

Since ϕ is none increasing in third and fifth argument?

$$\phi(M(z,Bz,kt), M(z,Bz,t), M(z,Bz,t), M(z,Bz,t), M(z,Bz,t) \geq 0$$

$\Rightarrow \quad M(z, Bz, kt) \geq M(Bz, z, t)$ {By Lemma}

$\Rightarrow \quad$ Bz = z

So we get

$Az = Bz = Sz = Tz = z$

Hence *z* is a common fixed point A, B, S, and T

Uniqueness: Let z and z' be two common fixed points of the maps A, B, S and T. Then

$Az = Bz = Tz = Sz = z$ and $Az' = Bz' = Tz' = Sz' = z'$

Using contractive condition, we get

$$\varphi(M(Az, Bz', kt), M(Sz, Tz', t), M(Sz, Az, t), M(Tz', Bz', kt), M(Tz', Az, t) \geq 0$$

$$\Rightarrow \varphi(M(z, z', kt), M(z, z', t), M(z, z, t), M(z', z', kt), M(z', z, t) \geq 0$$

$$\Rightarrow \phi(M(z, z', kt), M(z, z', t), M(z, z', t), M(z', z, t), M(z', z, t) \geq 0$$

Since ϕ is none increasing in third and fourth argument so

$$\Rightarrow \varphi(M(z, z', kt), M(z, z', t), M(z, z', t), M(z', z', t) \geq 0$$

$\Rightarrow M(z, z', kt) \geq M(z, z', t)$ {By Second conducting}

$\Rightarrow z = z'$ (By Lemma)

Hence z is a unique common fixed point maps A, B, S, T.

Corollary 3.1: Let A, B, S and T be self mappings of a complete fuzzy metric space $(X, M, *)$ with continuous t-norm defined by $a * b = \min\{a * b\}\{b\} \in [0,1]$ Satisfying I to III.

$$\varphi\left(M(Ax, By, kt), M(Sx, Ty, t), M(Sx, Ax, t), M(Tx, Ay, t)\right) \geq 0$$

then A, B, S and T have a unique common fixed point.

Corollary 3.2: Let A, B, S and T be self mappings of a complete fuzzy metric space $(X, M, *)$ with Continuous t-norm defined by $a * b = \min\{a * b\}\{b\} \in [0,1]$ Satisfying I, ii, iii, v of theorem3.1 and there exist some $k \in (0,1)$ such that for all $x, y \in X, t > 0$

$$\varphi\left(M(Ax, By, kt), M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, 2t), M(Ty, Ax, t)\right) \geq 0)$$

Then A, B, S and T have a unique common fixed point.

Theorem 3.2: Let $(X, M, *)$ be a complete fuzzy metric spaces. S and T have a common fixed point in X if and only if there exist a self mapping A of X such that the following condition is satisfied:

(i) $A(X) \subset T(X) \cap S(X)$

(ii) Pairs (A, S) and (A, T) are compatible of type (P)

(iii) $\exists$ $k \in (0,1)$ Such that for all $x, y \in X, t > 0$

$$\varphi(M(Ax, Ay, kt), M(Sx, Ty, t), M(Ax, Sx, t), M(Ty, Ay, kt), M(Ty, Ax, t)) \geq 0$$

Then A, B, S and T have a unique common fixed point.

Proof: We shown that the necessity of the conditions (I) - (iii). Suppose that *S* and *T* Have a common fixed point in *X*, say *z*. Then *so* = *z* = *Tz*.

Let $Ax = z$ *for* all $x \in X$. Then we have $A(X) \subset T(X) \cap S(X)$ and we know that [*A*, *S*] and [*A*, *T*] are compatible mapping of type (P), in fact *A S* = *SA and AT* =*T A*, and hence the conditions (I) and (ii) are satisfied.

For some p∈(0, 1), we get M (Ax, Ay, kt) =1 so

$$\varphi\big(M(Ax, Ay, kt), M(Sx, Ty, t), M(Ax, Sx, t),\ M(Ty, Ay, kt), M(Ty, Ax, t)\big) \geq 0 \text{ for all } x, y \in X, t > 0$$

And hence condition (iii) is satisfied.

Now for the sufficiency of conditions, let A = B in theorem3.1.then A, S and T

Have a common fixed point in X.

Corollary 3.3: Let A, B, S and T be self mappings of a complete fuzzy metric space $(X, M, *)$ with continuous t-norm Satisfying I to III of theorem 3.2 and

$$\varphi\big(\ M(Ax, By, kt), M(Sx, Ty, t), M(Sx, Ax, t), M(Tx, Ay, t)\big) \geq 0$$

Then A, B, S and T have a unique common fixed point.

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REFERENCES

1. Cho, Y.J. Sharma, B.K. and Sahu, D.R. 1995, Semi-compatibility and fixed points, Math. Japonica 42(1) 91-98.
2. Cho, on common fixed points in fuzzy metric spaces, Inter. Math. Forum, 1 (10) (2006), 471-479.
3. Deng Zi- Ke, Fuzzy pseudo metric spaces, J. Math. Anal. Appl.86 (1982), 74-95.
4. Erceg.M.A, Metric spaces in Fuzzy set theory, J. Math. Anal. Appl., 69(1979), 205-230.
5. A. George and P. Veeramani, on same results of analysis for fuzzy metric spaces, Fuzzy sets and systems, 90 (1997), 365-368.
6. Grabiec M., Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems, 27(1988), 385-389.
7. G. Jungck, Compatible mappings and common fixed points. Internat. J. Math. And Math. Sci., 9(4), (1986), 771-779.
8. Kramosil and J. Machalek, Fuzzy metric and statistical metric spaces, Kybernetika 11 (1975), 336-344.
9. M. Koireng and Y. Rohen, Common fixed point theorem of Compatibility of type P in fuzzymetric space, Int. Journal of Math. Analysis, Vol. 6, 2012, no. 4, 181 – 188.
10. O. Kaleva and S. Seikkala, on fuzzy metric spaces, Fuzzy Sets and Systems; 12(1984), 215-229.
11. Pathak, Chang, Cho, Fixed point theorems for compatible mappings oftype (P), Indian journal of Math. 36(2) (1994), 151-166.
12. B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math., 10(1960), 314- 334.
13. Bijedra Singh and M. S. Chauhan, Common fixed points of compatible maps in fuzzy metric spaces, Fuzzy sets and Systems, 115(2000), 471-475.
14. Singh B. and Chouhan, M.S. Common fixed points of compatible maps in fuzzy metric spaces, fuzzy sets and systems, 115, 471-475 (2000).
15. S. Sessa, On weak commutativity condition of mappings in fixed point onsiderations, Publ.Inst. Math. Besgrad, 32 (46) (1982) 149-153.
16. Zadeh L.A., Fuzzy sets, Inform and Control, 8 (1965), 338-353.

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