International Research Journal of Pure Algebra -3(7), 2013, 259-262
Available online through www.rjpa.info ISSN 2248–9037

Minimal Bi-ideals in Γ-Semirings

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(Received on: 12-07-13; Revised & Accepted on: 26-07-13)

ABSTRACT

In this paper we define a minimal bi-ideal and a 0-minimal bi-ideal of a Γ-semiring. Also we introduce the concepts of a bi-simple Γ-semiring and a 0-bi-simple Γ-semiring. Several characterizations of minimal bi-ideal , 0-minimal bi-ideal ,bi-simple Γ-semiring and 0-bi-simple Γ-semiring are furnished.

AMS Mathematics Subject Classification (2000): *16Y60, 16Y99.*

Key words: *Bi-ideal, minimal bi-ideal, 0-minimal bi-ideal, bi-simple Γ-semiring, 0-bi-simple Γ-semiring.*

1. INTRODUCTION

The notion of a Γ-semiring was introduced by Rao [12] as a generalization of a ring, a Γ-ring and a semiring. It is well known that ideals play an important role in any abstract algebraic structures. Characterizations of ideals in a semigroup were given by Lajos [8], while ideals in semirings were characterized by Iseki [4, 5].

The notion of a bi-ideal was first introduced for semigroups by Good and Hughes [2].The concept of a bi-ideal for a ring was given by Lajos [9]. Also in [10, 11] Lajos discussed some characterizations of bi-ideals in semigroups. Shabir Ali Batool in [13] gave some properties of bi-ideals in a semiring. Minimal bi-ideal for a semigroup was studied by Krgovic in [7] and for a Γ-semigroup by Iampan [3].

Hence in this paper we introduce the concepts of a minimal bi-ideal and 0-minimal bi-ideal of a Γ-semiring. Further discussed some of their characterizations. Also we introduce the notions of a bi-simple Γ-semiring and a 0-bi-simple Γ-semiring. Some properties of a bi-simple Γ-semiring and a 0-bi-simple Γ-semiring are also furnished.

2. PRELIMINARIES

First we recall some definitions of the basic concepts of Γ-semirings that we need in sequel. For this we follow Dutta and Sardar [1].

Definition 2.1: *Let S and Γ be two additive commutative semigroups. S is called a Γ-semiring if there exists a mapping* $S \times \Gamma \times S \longrightarrow S$ *denoted by* $a\alpha b$*; for all* $a, b \in S$ *and for all* $\alpha \in \Gamma$ *satisfying the following conditions:*
(i) $a\alpha(b + c) = (a\alpha b) + (a\alpha c)$
(ii) $(b + c)\alpha a = (b\alpha a) + (c\alpha a)$
(iii) $a(\alpha + \beta)c = (a\alpha c) + (a\beta c)$
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$; for all $a, b, c \in S$ *and,* $\beta \in \Gamma$.

Obviously, every semiring is a Γ-semiring.

Definition 2.2: *An element* $0 \in S$ *is said to be an absorbing zero if* $0\alpha a = 0 = a\alpha 0, a + 0 = 0 + a = a$; *for all* $a \in S$ *and for all* $\alpha \in \Gamma$.

Now onwards S denotes a Γ-semiring with absorbing zero unless otherwise stated.

Definition 2.3: *A non empty subset T of S is said to be a sub- Γ-semiring of S if (T, +) is a subsemigroup of (S, +) and* $a\alpha b \in T$*; for all* $a, b \in T$ *and for all* $\alpha \in \Gamma$.

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Definition 2.4: *A nonempty subset T of S is called a left (respectively right) ideal of S if T is a subsemigroup of (S, +) and* $x\alpha a \in T$ *(respectively* $a\alpha x \in T$*)* for all $a \in T, x \in S$ *and for all* $\alpha \in \Gamma$.

Definition 2.5: *If a nonempty subset T is both left and right ideal of S, then T is known as an ideal of S.*

For proofs of following result see [5].

Result 2.6: *For each nonempty subset X of S following statements hold.*
(i) $S\Gamma X$ *is a left ideal.*
(ii) $X\Gamma S$ *is a right ideal.*
(iii) $S\Gamma X\Gamma S$ *is an ideal of S.*

Result 2.7: *For* $a \in S$ *following statements hold.*
(i) $S\Gamma a$ *is a left ideal.*
(ii) $a\Gamma S$ *is a right ideal.*
(iii) $S\Gamma a\Gamma S$ *is an ideal of* S.

Now we give a definition of a bi-ideal.

Definition 2.8 [6]: *A nonempty subset* B *of* S *is a bi-ideal of* S *if* B *is a sub* Γ*-semiring of* S *and* $B\Gamma S\Gamma B \subseteq B$.

Example: Let N be the set of natural numbers and let $\Gamma = 2N$. Then N and Γ both are additive commutative semigroup. An image of a mapping $N \times \Gamma \times N \longrightarrow N$ is defined by $a\alpha b =$ product of a, α, b; for all $a, b \in S$ and $\alpha \in \Gamma$. Then S forms a Γ-semiring. $B = 4N$ is a bi-ideal of N.

Example: Consider a Γ-semiring $S = M_{2x2}(N_0)$, where N_0 denotes the set of natural numbers with zero and $\Gamma = S$. Define $A\alpha B$= usual matrix product of A, α and B; for all $A, \alpha, B \in S$. Then
$Q = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in N_0\right\}$ is a bi-ideal of a Γ-semiring S.

3. BI-SIMPLE Γ-SEMIRING

We begin with defining a bi-simple Γ-semiring and a 0-bisimple Γ-semiring.

Definition 3.1: *A* Γ*-semiring* S without zero is *a bi-simple* Γ*-semiring if* S *has no bi-ideal other than* S *itself.*

Definition 3.2: *If* Γ*-semiring* S *contains a zero element, then* S *is a 0-bi-simple Γ-semiring if* S *and* $\{0\}$ *are the only bi-ideals of* S.

Next theorem gives a characterization of a bi-simple Γ-semiring.

Theorem 3. 3: *If* S *is a* Γ*-semiring without zero, then* S *is a bi-simple* Γ*-semiring if and only if* $a\Gamma S\Gamma a = a$*, for all* $a \in S$.

Proof: Suppose that S is a bi-simple Γ-semiring. For any $a \in S$, $a\Gamma S\Gamma a$ is a sub Γ-semiring of S. By Result 2.7(ii) $a\Gamma S$ is a right ideal of S and hence $(a\Gamma S\Gamma a)\Gamma S\Gamma(a\Gamma S\Gamma a) = (a\Gamma S)\Gamma(a\Gamma S\Gamma a\Gamma S)\Gamma a \subseteq (a\Gamma S)\Gamma(a\Gamma S)\Gamma a$. Therefore $(a\Gamma S\Gamma a)\Gamma S\Gamma(a\Gamma S\Gamma a) \subseteq \big((a\Gamma S)\Gamma(a\Gamma S)\big)\Gamma a \subseteq a\Gamma S\Gamma a$, since $a\Gamma S$ is a right ideal of S. Thus $a\Gamma S\Gamma a$ is a bi-ideal of S by definition. $a\Gamma S\Gamma a \subseteq S$ and S is bi-simple Γ-semiring S imply $a\Gamma S\Gamma a = S$. Conversely, suppose that $a\Gamma S\Gamma a = S$. Let B be a bi-ideal of S. For any $b \in B$, $b\Gamma S\Gamma b = S$ by assumption. $S = b\Gamma S\Gamma b \subseteq B\Gamma S\Gamma B \subseteq B$ as $b \in B$ and B is a bi-ideal of S. Therefore $B = S$. Hence S is a bi-simple Γ semiring.

Theorem 3.4: *If* S *is a* Γ*-semiring* without zero, *then* S *is a bi-simple* Γ*-semiring if and only if* $(a)_b = S$*, for* $a \in S$.

Proof: Let S be a bi-simple Γ-semiring. For any $a \in S$, we have $(a)_b \subseteq S$. But S is bi-simple gives $(a)_b = S$. Conversely, let B be a bi-ideal of S. Then for any $a \in B$, $(a)_b = S$ by assumption. $S = (a)_b \subseteq B$. Therefore we have $B = S$ as $B \subseteq S$ and $S \subseteq B$.

Proof of following theorem is follows from above two theorems.

Theorem 3. 5: *If* S *is a* Γ*-semiring with zero, then following statements are equivalent*
(1) S is *a 0-bi-simple* Γ*-semiring*
(2) $a\Gamma S\Gamma a = S$*, for* $a \in S\backslash\{0\}$
(3) $(a)_b = S$*, for* $a \in S\backslash\{0\}$.

4. MINIMAL BI-IDEALS:

Definition 4.1: *Let S be a* Γ*-semiring. A bi-ideal B of S said to be a minimal bi-ideal of S if B does not contain any other proper bi-ideal of S.*

Definition 4.2: *Let S be a* Γ*-semiring with zero. A bi-ideal B of S is said to be 0-minimal bi-ideal if B does not contain any other proper non zero bi-ideal of S.*

Theorem 4.3: *Let S be a* Γ*-semiring, B be a bi-ideal and T be a sub-* Γ*-semiring of S. If T is a bi-simple with $T \cap B \neq \emptyset$, then $T \subseteq B$.*

Proof: Let T be a bi-simple sub Γ-semiring with $T \cap B \neq \emptyset$. Then $a \in T \cap B$. $a \in T$, $a\Gamma T\Gamma a$ is a bi-ideal of T and T is a bi-simple imply $a\Gamma T\Gamma a = T$, by Theorem 3.3. Therefore $T = a\Gamma T\Gamma a \subseteq B\Gamma T\Gamma B \subseteq B\Gamma S\Gamma B \subseteq B$, since B is a bi-ideal. Thus we get $T \subseteq B$.

Theorem 4.4: *Let S be a* Γ*-semiring with zero, B be a bi-ideal and T be a sub-*Γ*-semiring of S. If T is a 0-bi-simple with $T\setminus\{0\} \cap B \neq \emptyset$, then $T \subseteq B$.*

Proof: Let T be a bi-simple sub Γ-semiring with $T\setminus\{0\} \cap B \neq \emptyset$. Then $a \in T\setminus\{0\} \cap B$. $a \in T$ and T is a 0-quasi-simple imply $a\Gamma T\Gamma a = T$ by Theorem 3.5. Therefore $T = a\Gamma T\Gamma a \subseteq B\Gamma T\Gamma B \subseteq B\Gamma S\Gamma B \subseteq B$, since B is a bi-ideal. Thus we have $T \subseteq Q$.

Properties of a minimal bi-ideal of a Γ-semiring S are proved in the following theorems.

Theorem 4.5: *Let R be a minimal right ideal and L be a minimal left ideal of a* Γ*-semiring S* without zero, *then $R\Gamma L$ is a minimal bi-ideal of S.*

Proof: Let R be a minimal right and L be a minimal left ideal of S. Let $B = R\Gamma L$. Then $R\Gamma L$ is a bi-ideal of S. Let A be a bi-ideal of S such that $A \subseteq B$. By Result 2.6 $S\Gamma A$ is a left ideal and $A\Gamma S$ is a right ideal of S. We have $\Gamma A \subseteq S\Gamma B = S\Gamma R\Gamma L$, since $A \subseteq B = R\Gamma L$ and $S\Gamma R\Gamma L \subseteq L$ as L is a left ideal. Therefore we get $S\Gamma A \subseteq L$. Similarly we can show that $A\Gamma S \subseteq R$. $S\Gamma A \subseteq L$ and L is a minimal left ideal of S gives $S\Gamma A = L$. $A\Gamma S \subseteq R$ and R is a minimal right ideal of S imply $A\Gamma S = R$. Therefore $B = R\Gamma L = A\Gamma S\Gamma S\Gamma A \subseteq A\Gamma S\Gamma A \subseteq A$ as A is a bi-ideal. Hence $B \subseteq A$. Thus we get $B = A$, since $A \subseteq B$. This shows B is a minimal bi-ideal of S.

Theorem 4.6: *Let B be a bi-ideal of* a Γ-semiring S without zero. *Then B itself is a bi-simple* Γ*-semiring if and only if B is a minimal bi-ideal of S.*

Proof: As B is a bi-ideal of S,B is a sub-Γ-semiring of S by definition. Suppose B is a bi-simple Γ-semiring. Let A be a bi-ideal of S such that $A \subseteq B$. Hence $A\Gamma B\Gamma A \subseteq A\Gamma S\Gamma A \subseteq A$, since A is a bi-ideal of S. Therefore A is a bi-ideal of B. $A \subseteq B$, A is a bi-ideal of B and B is a bi-simple Γ-semiring imply $A = B$. Therefore B is a minimal bi-ideal of S.

Conversely, let B be a minimal bi-ideal of S. For any $a \in B$, $a\Gamma S\Gamma a$ is a bi-ideal of S. $a\Gamma S\Gamma a \subseteq B\Gamma S\Gamma B \subseteq B$. As B is a minimal bi-ideal of S, $a\Gamma S\Gamma a = B$. This shows B itself a bi-simple Γ-semiring by Theorem 3.3.

Theorem 4.7: *Let B be a bi-ideal of* Γ*-semiring S with zero. If B is a 0-bi-simple* Γ*-semiring, then B is a 0-minimal bi-ideal of S.*

Proof: As B is a bi-ideal of S,B is a sub Γ-semiring of S by definition. Suppose B is a 0-bi-simple Γ-semiring. Let $\{0\} \neq A$ be a bi-ideal of S such that $A \subseteq B$.

Hence $A\Gamma B\Gamma A \subseteq A\Gamma S\Gamma A \subseteq A$, since A is a bi-ideal of S. Therefore A is a bi-ideal of B. $A \subseteq B$, A is a bi-ideal of B and B is a 0-bi-simple Γ-semiring imply $A = B$.

Therefore B is a 0-minimal bi-ideal of S.

Theorem 4.8: *Let B be a bi-ideal of a* Γ*-semiring S with zero. If B is a 0-minimal bi-ideal of S then either $B\Gamma B \neq \{0\}$ or B is a 0-bi-simple* Γ*-semiring* .

Proof: Let B be a 0-minimal bi-ideal of S. For any $0 \neq a \in B$, $a\Gamma S\Gamma a$ is a bi-ideal of S. $a\Gamma S\Gamma a \subseteq B\Gamma S\Gamma B \subseteq B$. As B is a 0-minimal bi-ideal of S and $a\Gamma S\Gamma a \neq \{0\}$, $a\Gamma S\Gamma a = B$. This shows B itself a 0-bi-simple Γ-semiring by Theorem 3.5.

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Source of support: Nil, Conflict of interest: None Declared