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PRIMARY IDEALS IN QUASI-COMMUTATIVE TERNARY SEMIGEOUPS

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ABSTRACT

In this paper we study the structure of cancellative quasi-commutative primary ternary semigroups. In fact we prove that if T is a cancellative quasi-commutative ternary semigroup, then (1) S is a primary ternary semigroup (2) proper prime ideals in T are maximal and (3) semiprimary ideals in T are primary, are equivalent.

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Key Words: *Quasi-commutative, Primary Ideal, Semiprimary Ideal, Proper Prime Ideal.*

INTRODUCTION

Satyanarayana [3] initiated the study of the structure of commutative primary semigroups and in [4] he characterized commutative semigroups containing identity with either one of the following properties (i) every nonzero primary ideal is prime as well as maximal (ii) every nonzero primary ideal is prime (iii) every nonzero ideal is prime. Anjaneylu [1], the notions of one sided primary ideals and semiprimary ideals in an arbitrary semigroup are introduced and a primary decomposition theorem for duo semigroups is established. Also in that, the class of all semigroups in which proper primary ideals are prime as well as maximal is characterized as a subclass of the class of all semisimple semigroups. In this paper we study the structure of cancellative quasi-commutative primary ternary semigroups.

1. PRELIMINARIES

Let T be a ternary semigroup. T is said to be ***quasi-commutative*** provided for any $a, b, c \in$ T, there exists a natural number n such that $abc = b^n ac = bca = c^n ba = cab = a^n cb$. Let A be an ideal in T. We denote the intersection of all prime ideals containing A by $\sqrt{A}$. An ideal A in T is said to be ***left*** **(*lateral, right*)** ***primary*** proved (1) $< x > < y > < z > \subseteq$ A, and $y, z \notin$ A, $(x, z \notin$ A, $x, y \notin$ A$)$ imply $x \in \sqrt{A}$ ($y \in \sqrt{A}$, $z \in \sqrt{A}$) and (2) $\sqrt{A}$ is a prime ideal. An ideal A in T is said to be ***primary*** provided it is a left primary, a lateral primary ideal and a right primary ideal. An ideal A in T is said to be ***semiprimary*** provided $\sqrt{A}$ is a prime ideal. T is said to be **(*left, lateral, right, semi*)** ***primary*** provided every ideal in T is (left, lateral, right, semi) primary. It is note that every left (lateral, right) primary ideal is a semiprimary ideal. For undefined terms used in this paper, the reader is referred to [5]. Throughout the paper T denotes quasi-commutative ternary semigroup unless otherwise stated.

Theorem 1.1: Let T be a ternary semigroup with identity and let M be the unique maximal ideal in T. If $\sqrt{A}$ = M for some ideal A in T, then A is a primary ideal.

Proof: Let $<x> <y> < z > \subseteq$ A and $y, z \notin$ A. If $x \notin \sqrt{A}$ then $< x > \nsubseteq \sqrt{A}$ = M. Since M is the union of all proper ideals in T, we have $< x > =$ T $\Rightarrow y, z \in < x >$ and hence $< y > = <x><y>< z > \subseteq$ A. It is a contradiction. Therefore $x \in \sqrt{A}$. Clearly $\sqrt{A}$ = M is a prime ideal. Thus A is left primary. By symmetry it follows that A is lateral primary and right primary. Therefore A is a primary ideal.

Lemma 1.2: Let A be any ideal in T. Then $\sqrt{A} = \{x \in$ T: $x^n \in$ A for some odd $n \in$ N$\}$.

Proof: Write S = $\{x \in$ T: $x^n \in$ A for some natural number $n\}$. Let $x \in$ S. Then $x^n \in$ A for some odd natural number n. If P is any prime ideal containing A, then $x^n \in$ A $\subseteq$ P and hence $< x >^n \subseteq$ P. So $x \in$ P and thus $x \in \sqrt{A}$. Conversely if $x \in \sqrt{A}$ and $x \notin$ S, then $x^n \notin$ A for all natural number n. By using Zorn's Lemma we can show that there is a prime ideal P such that $x \notin$ P, a contradiction. So $x \in$ T. Therefore T = $\sqrt{A}$.

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2. QUASI COMMUTATIVE PRIMARY TERNARY SEMIGROUPS

We begin with the following.

Theorem 2.1: In a quasi commutative ternary semigroup T, an ideal A of T is left primary iff right primary.

Proof: Suppose that A is a left primary ideal. Let $abc \in A$ and $a \notin A$, $c \notin A$. Since S is a quasi commutative semigroup, we have for each $a, b, c \in T$, there exists a odd natural number n such that abc = $b^n ac = bca = c^n ba = cab = a^n cb$. So $abc = b^n ac \in A$ and $a \notin A$, $c \notin A$. Since A is left primary, we have $b^n \in \sqrt{A}$ and since $\sqrt{A}$ is a prime ideal, $b \in \sqrt{A}$. Therefore A is a lateral primary ideal. Similarly we can prove that if A is a right primary ideal then A is a left primary ideal.

Theorem 2.2: In a quasi commutative ternary semigroup T, an ideal A of T is lateral primary iff right primary.

Proof: Suppose that A is a lateral primary ideal. Let $abc \in A$ and $a \notin A$, $b \notin A$. Since S is a quasi commutative semigroup, we have for each $a, b, c \in T$, there exists a odd natural number n such that $abc = b^n ac = bca = c^n ba = cab = a^n cb$. So $abc = bca \in A$ and $a \notin A$, $b \notin A$. Since A is lateral primary, we have $c \in \sqrt{A}$. Therefore A is a right primary ideal. Similarly we can prove that if A is a right primary ideal then A is a lateral primary ideal.

Theorem 2.3: In a quasi commutative ternary semigroup T, an ideal A of T is right primary iff left primary.

Proof: Suppose that A is a right primary ideal. Let $abc \in A$ and $b \notin A$, $c \notin A$. Since S is a quasi commutative semigroup, we have for each $a, b, c \in T$, there exists a odd natural number n such that $abc = b^n ac = bca = c^n ba = cab = a^n cb$. So $abc = bca \in A$ and $b \notin A$, $c \notin A$. Since A is right primary, we have $a \in \sqrt{A}$. Therefore A is a left primary ideal. Similarly we can prove that if A is a left primary ideal then A is a right primary ideal.

Corollary 2.4: If A is an ideal of a quasi commutative ternary semigroup T, then the following are equivalent.
1. A is primary.
2. A is left primary.
3. A is lateral primary
4. A is right primary.

Definition 2.5: A ternary semigroup T is said to be ***left cancellative*** if for all $a, b, x, y \in T$, $abx = aby \Rightarrow x = y$.

Definition 2.6: A ternary semigroup T is said to be ***laterally cancellative*** if for all $a, b, x, y \in T$, $axb = ayb \Rightarrow x = y$.

Definition 2.7: A ternary semigroup T is said to be ***right cancellative*** if for all $a, b, x, y \in T$, $xab = yab \Rightarrow x = y$.

Definition 2.8: A ternary semigroup T is said to be ***cancellative*** if T is left cancellative, right cancellative and laterally cancellative.

Theorem 2.9: Let T be a ternary semigroup with identity. If (non-zero, assume this T has zero) proper prime ideals in T are maximal, then T is a primary ternary semigroup.

Proof: Since T contains identity, T has a unique maximal ideal M, which is the union of all proper ideals in T. If A is a (nonzero) proper ideal in T, then $\sqrt{A} = M$ and hence by theorem 1.1, A is a primary ideal. If T has zero and if $< 0 >$ is a prime ideal, then $< 0 >$ is primary and hence T is primary. If $< 0 >$ is not a prime ideal, then $\sqrt{< 0 >} = M$ and hence by theorem 1.1, $< 0 >$ is a primary ideal. Therefore T is a primary ternary semigroup.

Note 2.10: If the ternary semigroup T has no identity, then we remark that theorem 2.9, is not true even if the ternary semigroup has a unique maximal ideal.

Example 2.11: Let T = {a, b, 1} be the ternary semigroup under the multiplication given in the following table.

.	a	b	1
a	a	a	a
b	a	b	b
1	a	b	1

Now T is a primary ternary semigroup in which the prime ideal $< a >$ is not a maximal ideal.

Theorem 2.12: Let T be a right cancellative quasi commutative ternary semigroup. If T is a primary ternary semigroup or a ternary semigroup in which semiprimary ideals are primary, then for any primary ideal Q, $\sqrt{Q}$ is non maximal implies $Q = \sqrt{Q}$ is prime.

Proof: Since $\sqrt{Q}$ is non maximal, there exists an ideal A in T such that $\sqrt{Q} \subset A \subset T$. Let $a \in A \setminus \sqrt{Q}$ and $b, c \in \sqrt{Q}$. Now $Q \subseteq Q \cup < abc > \subseteq \sqrt{Q}$. This implies $\sqrt{Q} \subseteq \sqrt{(Q \cup < abc >)} \subseteq \sqrt{(\sqrt{Q})} = \sqrt{Q}$. Hence $\sqrt{(Q \cup < abc >)} = \sqrt{Q}$. Thus by hypothesis $Q \cup <abc>$ is a primary ideal. Let $s, t \in T \setminus A$. Then for some natural number n, $asbtc = s^n abtc = s^n abct \in Q \cup < abc >$. Since $a \notin \sqrt{Q} = \sqrt{(Q \cup < abc >)}$ and $Q \cup < abc >$ is a primary ideal, $sbtc \in Q \cup < abc >$. If $sbtc \in < abc >$ then $sbtc = rabtc$ for some $r \in T^1$ and hence by right cancellative property, we have $s = ra \in A$, a contradiction. Thus $sbtc \in Q$, which implies, since $s \notin Q$, $btc \in Q$ and hence $\sqrt{Q} = Q$. Therefore $Q = \sqrt{Q}$ and so Q is prime.

Theorem 2.13: Let T be a right cancellative quasi commutative ternary semigroup. If T is either a primary ternary semigroup of a ternary semigroup in which semiprimary ideals are primary, then proper prime ideals in T are maximal.

Proof: First we show that if P is a minimal prime ideal containing a principal ideal $< d >$, then P is a maximal ideal. Suppose P is not a maximal ideal.

Write $M = T \setminus P$ and $A = \{x \in T: xmn \in < d > \text{ for some } m, n \in M\}$.

Let $x \in A$, $s, t \in T$. $x \in A \Rightarrow xmn \in < d > \Rightarrow xmn = s_1 d t_1$ for some $s_1, t_1 \in T$.

Now $stxmn = st(xmn) = st(s_1dt_1) = (sts_1)dt_1 \in < d > \Rightarrow stx \in A$, similarly $sxt \in A$ and $xst \in A$. Therefore A is an ideal of T.

If $x \in A$, then $xmn \in < d > \subseteq P$. Since P is prime ideal and hence $x \in P$. So $A \subseteq P$.

Let $b \in P$ and suppose $N = \{b^k mn : m, n \in M$ and k is a nonnegative odd interger$\}$.

If $b^k mn, b^s pq, b^r uv \in N$ for $m, n, p, q, u, v \in M$ and $k, s,$ and r are nonnegative odd integers.

Then $(b^k mn)(b^s pq)(b^r uv) = b^{k+s+r} mnpquv \in N$.

Therefore N is a ternary subsemigroup of T containing M properly.

If $b \in P \Rightarrow bmn \in P \Rightarrow bmn \notin M$ and hence $bmn \in N$ and $bmn \notin M$.

Since P is a minimal prime ideal containing $< d >$, M is a maximal ternary subsemigroup not meeting $< d >$. Since N contains M properly, we have $N \cap < d > \neq \emptyset$.

So there exist a odd natural number k such that $b^k mn \in < d > \Rightarrow b^k \in A \Rightarrow b \in \sqrt{A}$.

Since P is prime, by theorem 2.19, P is semiprime and by theorem 2.23, $P = \sqrt{P}$.

Therefore $P \subseteq \sqrt{A} \Rightarrow P \subseteq \sqrt{A} \subseteq \sqrt{P} = P$. So $P = \sqrt{A}$.

By hypothesis A is a primary ideal. Since P is not a maximal ideal, we have by theorem 3.28, $\sqrt{A} = A \Rightarrow P = A$. Since $< d > \subseteq P$ and $< d^3 > \subseteq < d >$.

Therefore $< d^3 > \subseteq P$ and hence P is also a minimal prime ideal containing $< d^3 >$.

Let $B = \{y \in T: ymn \in < d^3 > \text{ for some } m, n \in M\}$. As before, we have $B = P$.

Since $d \in P = A = B$, we have $dmn = std^3$ for some $s, t \in T^1$.

Since T is a quasi commutative ternary semigroup, $dmn = m^p nd = std^3$ for some natural number p. By right cancellative property $m^p n = std^2$, a contradiction.

Therefore P is maximal ideal. Now if P is any proper prime ideal, then for any $d \in P$, $< d >$ is contained in a minimal prime ideal, which is maximal by the above and hence P is a maximal ideal.

Corollary 2.14: If T is a cancellative commutative ternary semigroup such that either T is a primary ternary semigroup or in T an ideal A is primary if and only if $\sqrt{A}$ is a prime ideal, then the proper prime ideals in T are maximal.

Proof: The proof of this corollary is a direct consequence of theorem 2.13.

Theorem 2.15: Let T be a right cancellative quasi commutative ternary semigroup with identity. Then the following are equivalent.

1. Proper prime ideals in T are maximal.
2. T is a primary ternary semigroup.
3. Semiprimary ideals in T are primary.
4. If x, y and z are not units in T, then there exists natural numbers n, m and p such that $x^n = yzs$, $y^m = xzt$ and $z^p = xyu$ for some $s, t, u \in T$.

Proof: Combining theorem 2.9, and 2.13, we have (1), (2) and (3) are equivalent.

(1) ⇒ (4): Assume (1). Since T contains identity, T has a unique maximal ideal M,

which is the only prime ideal in T. If x, y and z are not units.

If $< x > \nsubseteq M$ then $< x > = T \Rightarrow 1 \in < x > \Rightarrow x$ is a unit, a contradiction and hence $x \in M$, similarly $y, z \in M$. Therefore $\sqrt{< x >} = \sqrt{< y >} = \sqrt{< z >} = M$

$\Rightarrow y, z \in \sqrt{< x >}$, $x, z \in \sqrt{< y >}$ and $x, y \in \sqrt{< z >} \Rightarrow x^n = yzs$, $y^m = xzu$ and $z^p = xyu$ for some $s, t, u \in T$.

(4) ⇒ (2): Let A be any ideal in T and $xyz \in A$. Suppose that x, y, z are not units in T.

Let $y, z \notin A$, then $x^n = yzs \Rightarrow x^{n+2} = xxyzs \in A$. Therefore $x \in \sqrt{A}$.

Therefore A is left primary. Similarly A is lateral primary and right primary.

Therefore T is primary ternary semigroup.

Note 2.16: If T has 0, then the theorem 2.15 is true by assuming nonzero proper prime ideals are maximal.

Theorem 2.17: Let T be a right cancellative quasi commutative ternary semigroup not containing identity. Then the following are equivalent.

1. T is a primary ternary semigroup
2. Semiprimary ideals in T are primary
3. T has no proper prime ideals.
4. If $x, y \in T$, then there exists odd natural numbers n, m and p such that $x^n = yzs$, $y^m = xzt$ and $z^p = xyu$ for some s, t, $u \in T$.

Proof: (1) ⇒ (2): Since T is primary ternary semigroup, then its every ideal is primary. Therefore semiprimary ideal is also primary.

(2) ⇒ (3): Assume (2). By theorem 2.13, proper prime ideals of T are maximal and hence if P is any prime ideal, then P is maximal. Let $a, b, c \in T\backslash P$. Suppose $abc \notin T\backslash P \Rightarrow abc \in P \Rightarrow$ either $a \in P$ or $b \in P$ or $c \in P$, a contradiction. Therefore $abc \in T\backslash P$. Clearly T\P satisfies associative property. Therefore T\P is ternary semigroup. Let $a, b \in T\backslash P$. Then $aaT \nsubseteq P$ and hence $P \cup aaT = T \Rightarrow b \in aaT \Rightarrow b = aax$ for some $x \in T$. If $x \in P$, then $b \in P$, a contradiction. Therefore $aax = b$ has a solution in T\P. Similarly $yaa = b$ has a solution in T\P and hence T\P is a ternary group. Let e be the identity of the group T\P. Now e is an idempotent in T and since S is a right cancellative ternary semigroup, then e is a left identity and lateral identity of T. Since T is a quasi commutative ternary semigroup, idempotents in T are commute and hence e is the identity of T, a contradiction, since T has no identity. Therefore T has no proper prime ideals.

(3) ⇒ (4): Suppose T has no proper prime ideals. Then for any ideal A of T, $\sqrt{A} = T$. Let $x, y, z \in T$. Now $\sqrt{< x >} = \sqrt{< y >} = \sqrt{< z >} = T \Rightarrow y, z \in \sqrt{< x >}$, $x, z \in \sqrt{< y >}$ and $x, y \in \sqrt{< z >} \Rightarrow y^m, z^p \in < x >$, x^n, $z^p \in < y >$ and $x^n, y^m \in < z >$ for some odd natural numbers $n, m, p \Rightarrow x^n = yzs$, $y^m = xzu$ and $z^p = xyu$ for some $s, t, u \in T$.

(4) ⇒ (1): Let A be any ideal of T. Let $xyz \in A$, Suppose that x, y, z are not units in T, then $x^n = yzs \Rightarrow x^{n+2} = xxyzs \in A \Rightarrow x \in \sqrt{A}$. Therefore A is left primary. Since T is quasi commutative ternary semigroup and hence A is lateral primary and right primary. Therefore A is primary and hence T is a primary ternary semigroup. This completes the proof of the theorem.

Theorem 2.18: Let T be a right cancellative quasi commutative ternary semigroup. Then the following are equivalent.

1. T is a primary ternary semigroup.
2. Semiprimary ideals in T are primary.
3. Proper prime ideals in T are maximal.

Proof: The proof of this theorem is a direct consequence of theorem 2.15, and 2.17.

Corollary 2.19: Let T be a cancellative commutative ternary semigroup. Then T is a primary ternary semigroup if and only if proper prime ideals in T are maximal. Furthermore T has no idempotents except identity, if it exists.

Proof: The proof of this corollary is a direct consequence of theorem 2.18.

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