In this paper we study the structure of cancellative quasi-commutative primary ternary semigroups. In fact we prove that if $T$ is a cancellative quasi-commutative ternary semigroup, then (1) $S$ is a primary ternary semigroup (2) proper prime ideals in $T$ are maximal and (3) semiprimary ideals in $T$ are primary, are equivalent.

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1. PRELIMINARIES

Let $T$ be a ternary semigroup. $T$ is said to be quasi-commutative provided for any $a, b, c \in T$, there exists a natural number $n$ such that $a b c = b^n a = c^n b = c a b = a^n c b$. Let $A$ be an ideal in $T$. We denote the intersection of all prime ideals containing $A$ by $\sqrt{A}$. An ideal $A$ in $T$ is said to be left (lateral, right) primary proved (1) $x < y < z \subseteq A$ and $y, z \not\in A$, (2) $\sqrt{A}$ is a prime ideal. An ideal $A$ in $T$ is said to be primary provided it is a left primary, a lateral primary ideal and a right primary ideal. An ideal $A$ in $T$ is said to be semiprimary provided $\sqrt{A}$ is a prime ideal. $T$ is said to be (left, lateral, right, semi) primary provided every ideal in $T$ is (left, lateral, right, semi) primary. It is note that every left (lateral, right) primary ideal is a semiprimary ideal. For undefined terms used in this paper, the reader is referred to [5]. Throughout the paper $T$ denotes quasi-commutative ternary semigroup unless otherwise stated.

Theorem 1.1: Let $A$ be a ternary semigroup with identity and let $M$ be the unique maximal ideal in $T$. If $\sqrt{A} = M$ for some ideal $A$ in $T$, then $A$ is a primary ideal.

Proof: Let $x < y < z \subseteq A$ and $y, z \not\in A$. If $x \not\in \sqrt{A}$ then $x < y < z \subseteq A$. Since $M$ is the union of all proper ideals in $T$, we have $x = T \Rightarrow y, z \in x$ and hence $y > x > y < z \subseteq A$. It is a contradiction. Therefore $x \in \sqrt{A}$.

Lemma 1.2: Let $A$ be any ideal in $T$. Then $\sqrt{A} = \{x \in T: x^n \in A \text{ for some odd } n \in N\}$.

Proof: Write $S = \{x \in T: x^n \in A \text{ for some odd natural number } n\}$. Let $x \in S$. Then $x^n \in A$ for some odd natural number $n$. If $P$ is any prime ideal containing $A$, then $x^n \in A \subseteq P$ and hence $x > x^n \subseteq P$. So $x \in P$ and thus $x \in \sqrt{A}$. Conversely if $x \in \sqrt{A}$ and $x \not\in S$, then $x^n \not\in A$ for all natural number $n$. By using Zorn’sLemma we can show that there is a prime ideal $P$ such that $x \not\in P$, a contradiction. So $x \in T$. Therefore $T = \sqrt{A}$.
2. QUASI COMMUTATIVE PRIMARY TERNARY SEMIGROUPS

We begin with the following.

Theorem 2.1: In a quasi commutative ternary semigroup T, an ideal A of T is left primary iff right primary.

Proof: Suppose that A is a left primary ideal. Let \( abc \in A \) and \( a \notin A, c \notin A \). Since S is a quasi commutative semigroup, we have for each \( a, b, c \in T \), there exists a odd natural number \( n \) such that \( abc = b^n ac = bca = b^n ba = cab = a^n cb \). So \( abc = b^n ac \in A \) and \( a \notin A, c \notin A \). Since A is left primary, we have \( b^n \in \sqrt{A} \) and since \( \sqrt{A} \) is a prime ideal, \( b \in \sqrt{A} \). Therefore A is a lateral primary ideal. Similarly we can prove that if A is a right primary ideal then A is a left primary ideal.

Theorem 2.2: In a quasi commutative ternary semigroup T, an ideal A of T is lateral primary iff right primary.

Proof: Suppose that A is a lateral primary ideal. Let \( abc \in A \) and \( a \notin A, b \notin A \). Since S is a quasi commutative semigroup, we have for each \( a, b, c \in T \), there exists a odd natural number \( n \) such that \( abc = b^n ac = bca = c^n ba = cab = a^n cb \). So \( abc = bca \in A \) and \( a \notin A, b \notin A \). Since A is lateral primary, we have \( c \in \sqrt{A} \). Therefore A is a right primary ideal. Similarly we can prove that if A is a right primary ideal then A is a lateral primary ideal.

Theorem 2.3: In a quasi commutative ternary semigroup T, an ideal A of T is right primary iff left primary.

Proof: Suppose that A is a right primary ideal. Let \( abc \in A \) and \( b \notin A, c \notin A \). Since S is a quasi commutative semigroup, we have for each \( a, b, c \in T \), there exists a odd natural number \( n \) such that \( abc = b^n ac = bca = c^n ba = cab = a^n cb \). So \( abc = bca \in A \) and \( b \notin A, c \notin A \). Since A is right primary, we have \( a \in \sqrt{A} \). Therefore A is a left primary ideal. Similarly we can prove that if A is a left primary ideal then A is a right primary ideal.

Corollary 2.4: If A is an ideal of a quasi commutative ternary semigroup T, then the following are equivalent.
1. A is primary.
2. A is left primary.
3. A is lateral primary.
4. A is right primary.

Definition 2.5: A ternary semigroup T is said to be **left cancellative** if for all \( a, b, x, y \in T \), \( abx = a by \Rightarrow x = y \).

Definition 2.6: A ternary semigroup T is said to be **laterally cancellative** if for all \( a, b, x, y \in T \), \( axb = ayb \Rightarrow x = y \).

Definition 2.7: A ternary semigroup T is said to be **right cancellative** if for all \( a, b, x, y \in T \), \( xab = yab \Rightarrow x = y \).

Definition 2.8: A ternary semigroup T is said to be **cancellative** if T is left cancellative, right cancellative and laterally cancellative.

Theorem 2.9: Let T be a ternary semigroup with identity. If (non-zero, assume this T has zero) proper prime ideals in T are maximal, then T is a primary ternary semigroup.

Proof: Since T contains identity, T has a unique maximal ideal M, which is the union of all proper ideals in T. If A is a (nonzero) proper ideal in T, then \( \sqrt{A} = M \) and hence by theorem 1.1, A is a primary ideal. If T has zero and if \(<0> \) is a prime ideal, then \(<0> is primary and hence T is primary. If \(<0> is not a prime ideal, then \( \sqrt{<0>} = M \) and hence by theorem 1.1, \(<0> is a primary ideal. Therefore T is a primary ternary semigroup.

Note 2.10: If the ternary semigroup T has no identity, then we remark that theorem 2.9, is not true even if the ternary semigroup has a unique maximal ideal.

Example 2.11: Let \( T = \{a, b, 1\} \) be the ternary semigroup under the multiplication given in the following table.

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Now \( T \) is a primary ternary semigroup in which the prime ideal \(<a>\) is not a maximal ideal.

**Theorem 2.12:** Let \( T \) be a right cancellative quasi-commutative ternary semigroup. If \( T \) is a primary ternary semigroup or a ternary semigroup in which semiprimary ideals are primary, then for any primary ideal \( Q \), \( \sqrt{Q} \) is non maximal implies \( Q = \sqrt{Q} \) is prime.

**Proof:** Since \( \sqrt{Q} \) is non maximal, there exists an ideal \( A \) in \( T \) such that \( \sqrt{Q} \subseteq A \subseteq T \). Let \( a \in A \setminus \sqrt{Q} \) and \( b, c \in \sqrt{Q} \).

Now \( Q \subseteq Q \cup <abc> \subseteq \sqrt{Q} \). This implies \( \sqrt{Q} \subseteq \sqrt{Q} \cup <abc> \subseteq \sqrt{\sqrt{Q}} = \sqrt{Q} \). Hence \( \sqrt{Q} \cup <abc> = \sqrt{Q} \).

Thus by hypothesis \( Q \cup <abc> \) is a primary ideal. Let \( s, t \in T \). Then for some natural number \( n \), \( sbtc = s^\prime abc = s^\prime abct \in Q < abc > \).

Since \( a \notin \sqrt{Q} = \sqrt{(Q \cup <abc>)} \) and \( Q \cup <abc> \) is a primary ideal, \( sbtc \in Q \cup <abc> \). If \( sbtc \in <abc> \) then \( sbtc = rabc \) for some \( r \in T \) and hence by right cancellative property, we have \( s = ra \in A \), a contradiction. Thus \( sbtc \in Q \), which implies, since \( s \notin Q \), \( btc \in Q \) and hence \( \sqrt{Q} = Q \). Therefore \( Q = \sqrt{Q} \) and so \( Q \) is prime.

**Theorem 2.13:** Let \( T \) be a right cancellative quasi-commutative ternary semigroup. If \( T \) is either a primary ternary semigroup of a ternary semigroup in which semiprimary ideals are primary, then proper prime ideals in \( T \) are maximal.

**Proof:** First we show that if \( P \) is a minimal prime ideal containing a principal ideal \(<d>\), then \( P \) is a maximal ideal. Suppose \( P \) is not a maximal ideal.

Write \( M = T \setminus P \) and \( A = \{x \in T : xmn < d \} \) for some \( m, n \in M \).

Let \( x \in A, s, t \in T \). \( x \in A \Rightarrow xmn < d \Rightarrow xmn = sdt_1 \) for some \( s_1, t_1 \in T \).

Now \( stxmn = st(xmn) = st(sdt_1) = (st(s_1))d_1t_1 \in <d> \Rightarrow stx \in A \), similarly \( stx \in A \) and \( stx \in A \). Therefore \( A \) is an ideal of \( T \).

If \( x \in A \), then \( xmn < d \) \( \subseteq P \). Since \( P \) is prime ideal and hence \( x \in P \). So \( A \subseteq P \).

Let \( b \in P \) and suppose \( N = \{b^kmn : m, n \in M \) and \( k \) is a nonnegative odd integer\}.

If \( b^kmn, b^kpq, b^kuv \in N \) for \( m, n, p, q, u, v \in M \) and \( k, s, \) and \( r \) are nonnegative odd integers.

Then \((b^k)^{(m)(n)}(b^k)^{(pq)}(b^k)^{(uv)}) = b^{k+ir+mpq+rv} \in N \).

Therefore \( N \) is a ternary subsemigroup of \( T \) containing \( M \) properly.

If \( b \in P \Rightarrow bmn \in P \Rightarrow bmn \in M \) and hence \( bmn \in N \) and \( bmn \in M \).

Since \( P \) is a minimal prime ideal containing \(<d>\), \( M \) is a maximal ternary subsemigroup not meeting \(<d>\). Since \( N \) contains \( M \) properly, we have \( N \cap <d> \neq \emptyset \).

So there exist a odd natural number \( k \) such that \( b^kmn \in <d> \Rightarrow b^k \in A \Rightarrow b \in \sqrt{A} \).

Since \( P \) is prime, by theorem 2.19, \( P \) is semiprime and by theorem 2.23, \( P = \sqrt{P} \).

Therefore \( P \subseteq \sqrt{A} \Rightarrow P \subseteq \sqrt{A} \subseteq \sqrt{P} = P \). So \( P = \sqrt{A} \).

By hypothesis \( A \) is a primary ideal. Since \( P \) is not a maximal ideal, we have by theorem 3.28, \( \sqrt{A} = A \Rightarrow P = A \). Since \(<d> \subseteq P \) and \(<d^3> \subseteq <d> \).

Therefore \(<d^3> \subseteq P \) and hence \( P \) is also a minimal prime ideal containing \(<d^3>\).

Let \( B = \{y \in T : ymn < d^3 \} \) for some \( m, n \in M \}. \) As before, we have \( B = P \).

Since \( d \in P \Rightarrow A = B \), we have \( dmn = std^3 \) for some \( s, t \in T \).

Since \( T \) is a quasi-commutative ternary semigroup, \( dmn = m^rnda = std^3 \) for some natural number \( p \). By right cancellative property \( pmn = std^3 \), a contradiction.

Therefore \( P \) is maximal ideal. Now if \( P \) is any proper prime ideal, then for any \( d \in P, <d> \) is contained in a minimal prime ideal, which is maximal by the above and hence \( P \) is a maximal ideal.
Corollary 2.14: If T is a cancellative commutative ternary semigroup such that either T is a primary ternary semigroup or in T an ideal A is primary if and only if \( \sqrt{A} \) is a prime ideal, then the proper prime ideals in T are maximal.

Proof: The proof of this corollary is a direct consequence of theorem 2.13.

Theorem 2.15: Let T be a right cancellative quasi commutative ternary semigroup with identity. Then the following are equivalent.
1. Proper prime ideals in T are maximal.
2. T is a primary ternary semigroup.
3. Semiprimary ideals in T are primary.
4. If \( x, y \) and \( z \) are not units in T, then there exists natural numbers \( n, m \) and \( p \) such that \( x^n = yzs \), \( y^m = xzt \) and \( z^p = xuy \) for some \( s, t, u \in T \).

Proof: Combining theorem 2.9, and 2.13, we have (1), (2) and (3) are equivalent.

(1) \( \Rightarrow \) (4): Assume (1). Since T contains identity, T has a unique maximal ideal M, which is the only prime ideal in T. If \( x, y \) and \( z \) are not units.

If \( \langle x \rangle \not\subseteq M \) then \( \langle x \rangle = \{ x \} \Rightarrow 1 \in \langle x \rangle \Rightarrow x \) is a unit, a contradiction and hence \( x \in M \). Similarly \( y, z \in M \). Therefore \( \sqrt{\langle x \rangle} = \sqrt{\langle y \rangle} = \sqrt{\langle z \rangle} = M \).

\[ y, z \in \sqrt{\langle x \rangle}, \ x, z \in \sqrt{\langle y \rangle} \ and \ x, y \in \sqrt{\langle z \rangle} \Rightarrow x^n = yzs, \ y^m = xzu \ and \ z^p = xuy \ \text{for some} \ s, t, u \in T. \]

(4) \( \Rightarrow \) (2): Let A be any ideal in T and \( xyz \in A \). Suppose that \( x, y, z \) are not units in T.

Let \( y, z \in A \), then \( x^n = yzs \Rightarrow x^{n+2} = xxyzs \in A \). Therefore \( x \in \sqrt{A} \).

Therefore A is left primary. Similarly A is lateral primary and right primary.

Therefore T is primary ternary semigroup.

Note 2.16: If T has 0, then the theorem 2.15 is true by assuming nonzero proper prime ideals are maximal.

Theorem 2.17: Let T be a right cancellative quasi commutative ternary semigroup not containing identity. Then the following are equivalent.
1. T is a primary ternary semigroup
2. Semiprimary ideals in T are primary
3. T has no proper prime ideals.
4. If \( x, y \in T \), then there exists odd natural numbers \( n, m \) and \( p \) such that \( x^n = yzs \), \( y^m = xzt \) and \( z^p = xuy \) for some \( s, t, u \in T \).

Proof: (1) \( \Rightarrow \) (2): Since T is primary ternary semigroup, then its every ideal is primary. Therefore semiprimary ideal is also primary.

(2) \( \Rightarrow \) (3): Assume (2). By theorem 2.13, proper prime ideals of T are maximal and hence if \( P \) is any prime ideal, then \( P \) is maximal. Let \( a, b, c \in TP \). Suppose \( abc \notin TP \Rightarrow abc \in P \Rightarrow \) either \( \not\in P \) or \( \not\in P \) or \( \not\in P \), a contradiction. Therefore \( abc \in TP \). Clearly TP satisfies associative property. Therefore \( TP \) is ternary semigroup. Let \( a, b \in TP \). Then \( aaT \not\subseteq P \) and hence \( P \cup aaT = T \Rightarrow b \in aaT \Rightarrow b = aax \) for some \( x \in T \). If \( x \in P \), then \( b \in P \), a contradiction. Therefore \( aax = b \) has a solution in TP. Similarly \( yaa = b \) has a solution in TP and hence \( TP \) is a ternary group. Let \( e \) be the identity of the group \( TP \). Now \( e \) is an idempotent in \( T \) and since \( S \) is a right cancellative ternary semigroup, then \( e \) is a left identity and lateral identity of T. Since T is a quasi commutative ternary semigroup, idempotents in T are commute and hence e is the identity of T, a contradiction, since T has no identity. Therefore T has no proper prime ideals.

(3) \( \Rightarrow \) (4): Suppose T has no proper prime ideals. Then for any ideal A of T, \( \sqrt{A} = T \). Let \( x, y, z \in T \). Now \( \sqrt{\langle x \rangle} = \sqrt{\langle y \rangle} = \sqrt{\langle z \rangle} = T \Rightarrow y, z \in \sqrt{\langle x \rangle}, \ x, z \in \sqrt{\langle y \rangle} \ and \ x, y \in \sqrt{\langle z \rangle} \Rightarrow y^n, z^p \in \langle x \rangle, x^m, z^p \in \langle y \rangle \ and \ x^n, y^m \in \langle z \rangle \) for some odd natural numbers \( n, m, p \). If \( x^n = yzs, \ y^m = xzu \ and \ z^p = xuy \ \text{for some} \ s, t, u \in T. \)

(4) \( \Rightarrow \) (1): Let A be any ideal of T. Let \( xyz \in A \). Suppose that \( x, y, z \) are not units in T, then \( x^n = yzs \Rightarrow x^{n+2} = xxyzs \in A \Rightarrow x \in \sqrt{A} \). Therefore A is left primary. Since T is quasi commutative ternary semigroup and hence A is lateral primary and right primary. Therefore A is primary and hence T is a primary ternary semigroup. This completes the proof of the theorem.
Theorem 2.18: Let T be a right cancellative quasi commutative ternary semigroup. Then the following are equivalent.
1. T is a primary ternary semigroup.
2. Semiprimary ideals in T are primary.
3. Proper prime ideals in T are maximal.

Proof: The proof of this theorem is a direct consequence of theorem 2.15, and 2.17.

Corollary 2.19: Let T be a cancellative commutative ternary semigroup. Then T is a primary ternary semigroup if and only if proper prime ideals in T are maximal. Furthermore T has no idempotents except identity, if it exists.

Proof: The proof of this corollary is a direct consequence of theorem 2.18.

REFERENCES

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