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ON STRONG FUZZY GRAPHS AND PROPERTIES

D. Venugopalam[1], N. Naga Maruthi Kumari[2*] and M. Vijaya Kumar[3]

[1]*Govt. Pingle College, Waddepalli, Hanamkonda, India*

[2]*ATRIA Institute of Technology, Anand Nagar, Bangalore, India*

[3]*Department of Mathematics, Vaagdevi College of Engineering, Warangal, (A.P.), India*

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ABSTRACT

This paper discuss on strong fuzzy graphs and properties. The connectedness of isomorphic fuzzy graphs discussed. The image of strong of fuzzy graph under isomorphism, a weak isomorphism is also studied.

Key Words: *Fuzzy relation, Fuzzy graphs, Strong Fuzzy graphs, Complement, Isomorphism.*

1. INTRODUCTION

Rosenfeld (1975) introduced the notion of fuzzy graph and several fuzzy analogs of graph theoretic concepts, such as paths, cycles and connectedness. Fuzzy models are becoming useful because of their aim in reducing the difference between the traditional numerical models used in Engineering and Science and Symbolic models used in expert system. Bhattacharya (1987) and Bhutani (1987) investigated the concept of fuzzy automorphism groups. This paper discusses some properties of isomorphic fuzzy graphs with reference to strong arcs in fuzzy graphs, strong fuzzy graphs and also about complement of a fuzzy graph.

Definition 1.1: The $G = G_1 [G_2]$ of two fuzzy graphs $G_i = (V_i, X_i)$ is defined [5] as a fuzzy graph $(\sigma_1 [\sigma_2], \mu_1 [\mu_2])$ on $G = (V, X^0)$ where $V = V_1 \times V_2$, $X^0 = X \cup X'$. Here X is as defined in Definition 1.5, $X' = \{((u_1, w_1), (v_1, w_2))| (u_1, v_1) \in X_1, w_1 \neq w_2\}$. Fuzzy sets $\sigma_1 [\sigma_2] = (\sigma_1 \times \sigma_2)$ on $V_1 \times V_2$ and $\mu_1 [\mu_2] = \mu_1 \times \mu_2$ on X and on X', $\mu_1 [\mu_2]$ is defined as $\mu_1[\mu_2] ((u_1, w_1), v_1, w_2)) = \mu_1(u_1, v_1) \Lambda \sigma_2(w_1) \Lambda \sigma_2(w_1) \Lambda \sigma_2(w_2)$.

Definition 1.2: The union $G = G_1 \cup G_2$ of two fuzzy graphs Gi = (Vi, Xi), i = 1, 2 is defined [5] as a fuzzy graph $(\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ on $G = (V_1 \cup V_2, X_1 \cup X_2)$ as follows : $(\sigma_1 \cup \sigma_2) (u) = \sigma_1(u)$ if $u \in V_1 \setminus V_2 = \sigma_2(u)$ if $u \in V_2 \setminus V_1$, and $(\sigma_1 \cup \sigma_2) (u) = \sigma_1(u) \vee \sigma_2(u)$ if $u \in V_1 \cap V_2$. Also $(\mu_1 \cup \mu_2) (u, v) = \mu_1 (u, v)$ if $(u, v) \in X_1 \setminus X_2 = \mu_2 (u, v)$ if $(u, v) \in X_2 \setminus X_1$, and $(\mu_1 \cup \mu_2) (u, v) = \mu_1 (u, v) \vee \mu_2 (u, v)$ if $(u, v) \in X_1 \cap X_2$.

Definition 1.3: Let $G = G_1 + G_2 = (V_1 \cup V_2, X_1 \cup X_2 \cup X')$ denote the join [5] of two fuzzy graph G1 = (V_1, X_1) and $G_2 = (V_2, X_2)$, where we assume that $V_1 \cap V_2 = \emptyset$ and X' is the set of all edges joining vertices of V_1 with the vertices of V_2. Define fuzzy sets $\sigma_1 + \sigma_2$ of $V_1 \cup V_2$ and $\mu_1 + \mu_2$ of $X_1 \cup X_2 \cup X'$ as follows: $(\sigma_1 + \sigma_2) (u) = \sigma_1 \cup \sigma_2(u)$ $\forall u \in V_1 \cup V_2$; $(\mu_1 + \mu_2) (u, v) = (\mu_1 \cup \mu_2) (u, v)$ if $(u, v) \in X_1 \cup X_2$ and $(\mu_1 + \mu_2) (u, v) = \sigma_1(u) \Lambda \sigma_2(v)$ if $(u, v) \in X'$.

Definition 1.4: A fuzzy graph with S as the underlying set is a pair G: (σ , μ) where $\sigma : S \to [0,1]$ is a fuzzy subset, $\mu : S\times S \to [0,1]$ is a fuzzy relation on the fuzzy subset σ, such that $\mu (x ,y) \leq \sigma (x) \wedge \sigma (y)$ for all $x, y \in S$, where $\wedge$ stands for minimum. The underlying crisp graph of the fuzzy graph G: (σ, μ) is denoted as $G^*: (\sigma^*, \mu^*)$ where $\sigma^* = supp (\sigma) = \{u\in S /\sigma (u)>0\}$, $\mu^* = supp(\mu)= \{(u, v) \in S \times S/ \mu (u, v)>0\}$.

Throughout this paper G:(σ, μ) and G':(σ', μ') are taken to be the fuzzy graphs with underlying sets S and S' respectively.

Definition 1.5: A path ρ in a fuzzy graph G: (σ, μ) is a sequence of distinct nodes $v_0, v_1, v_2,\ldots, v_n$ such that $\mu (v_{i-1}, v_i) > 0, 1 \leq i \leq n$. Here 'n' is called the length of the path. The consecutive pairs (v_{i-1}, v_i) are called arcs of the path.

Corresponding author: *N. Naga Maruthi Kumari[2]*
ATRIA Institute of Technology, Ananad Nagar, Bangalore, India

Definition 1.6: If u, v are nodes in G and if they are connected by means of a path then the strength of that path is defined as $\overset{n}{\underset{i=1}{\wedge}} \mu(v_{i-1}, v_i)$ i.e., it is the strength of the weakest arc. If u, v are connected by means of paths of length 'k' then μ^k (u, v) is defined as $\mu^k(u, v) = \sup\{\mu(u, v_1) \wedge \mu(v_1, v_2) \wedge \mu(v_2, v_3) \ldots \wedge \mu(v_{k-1}, v) / u, v_1, v_2, \ldots, v_{k-1}, v \in S\}$. If u, v $\in$S the strength of connectedness between u and v is $\mu^\infty(u, v) = \sup\{\mu^k(u, v) / k = 1,2,3,\ldots\}$.

Definition 1.7: A fuzzy graph G is connected if $\mu^\infty(u, v) > 0$ for all u, v $\in \sigma^*$.

Definition 1.8: A fuzzy graph G is said to be a strong fuzzy graph if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all (x, y) in μ^*.

Definition 1.9*:* A fuzzy graph G is said to be a complete fuzzy graph if $\mu(x, y) = \sigma(x) \wedge \sigma(y)$ for all x, y in σ^*.

Definition 1.10: An arc (x, y) is said to be a strong arc if $\mu(x, y) \geq \mu^\infty(x, y)$. A node x, is said to be an isolated node if $\mu(x, y) = 0 \ \forall \ y \neq x$.

Definition 1.11: A homomorphism of fuzzy graphs h: G $\to$ G' is a map h: S $\to$S' which satisfies$\sigma(x) \leq \sigma'(h(x))$ for all x$\in$S and $\mu(x, y) \leq \mu'(h(x), h(y))$ for all x, y $\in$S.

Definition 1.12: A weak isomorphism h: G $\to$G' is a map, h: S $\to$S' which is a bijective homomorphism that satisfies $\sigma(x) = \sigma'(h(x))$ for all x $\in$S.

Definition 1.13: A co-weak isomorphism h: G $\to$G' is a map, h: S $\to$S' which is a bijective homomorphism that satisfies $\mu(x, y) = \mu'(h(x), h(y))$ for all x, y $\in$S.

Definition 1.14: An isomorphism h: G $\to$G' is a map, h: S $\to$S' which is bijective that satisfies
$\sigma(x) = \sigma'(h(x))$ for all x $\in$S.
$\mu(x, y) = \mu'(h(x), h(y))$ for all x ,y $\in$S, and we denote G $\cong$ G'.

Definition 1.15: Let G :(σ, μ) be a fuzzy graph .The complement of G is defined as $\overline{G} : (\sigma, \overline{\mu})$ where $\overline{\mu}(x, y) = \sigma(x) \wedge \sigma(y) - \mu(x, y) \forall x, y \in S$. When G is a fuzzy graph, $\overline{G} : (\sigma, \overline{\mu})$ is also a fuzzy graph.

Definition 1.16: Given a fuzzy graph G: (σ,μ), with the underlying set S, the order of G is defined and denoted as $p = \sum_{x \in S} \sigma(x)$ and size of G is defined and denoted as

$$q = \sum_{x, y \in S} \mu(x, y).$$

2. MAIN RESULTS (STRONG FUZZY GRAPHS)

Theorem 2.1: If G is a connected, strong fuzzy graph then every arc in G is a strong arc.

Theorem 2.2: If G is to G' then G is a strong fuzzy graph iff G' is also a strong fuzzy graph.

Theorem 2.3: If G is co -weak with a strong fuzzy graph G' then G is also a strong fuzzy graph.

Theorem 2.4: G :(σ, μ) is a strong fuzzy graph iff G :(σ, μ) is also a strong fuzzy graph.

Theorem 2.5: G :(σ, μ) is a complete fuzzy graph iff G :(σ, μ) is an isolated fuzzy graph.

Theorem 2.6: If G_1 and G_2 are strong fuzzy graphs, then G_1 x G_2, G_1 [G_2] and $G_1 + G_2$ are also strong.

Proof: By Definition

$$\begin{aligned}(\mu_1 \times \mu_2)((u, u_2)(u, v_2)) &= \sigma_1(u) \wedge \mu_2(u_2, v_2) = \sigma_1(u) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2)\\ &= ((\sigma_1 \times \sigma_2)(u, u_2)) \wedge ((\sigma_1 \times \sigma_2)(u, v_2))\end{aligned}$$

$$\begin{aligned}\mu_1[\mu_2]((u_1, w_1), (v_1, w_2)) &= \mu_1(u_1, v_1) \wedge \sigma_2(w_1) \wedge \sigma_2(w_2)\\ &= \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(w_1) \wedge \sigma_2(w_2)\\ &= ((\sigma_1 \times \sigma_2)(u_1, w_1)) \wedge ((\sigma_1 \times \sigma_2)(v_2, w_2))\end{aligned}$$

$$(\mu_1 + \mu_2)(u, v) = \sigma_1(u) \wedge \sigma_2(v) \text{ on } X'$$

All this shows that $G_1 \times G_2$, $G_1[G_2]$ and $G_1 + G_2$ are also strong fuzzy graph.

Definition 2.7: We say a fuzzy subgraph $H = (\sigma, \tau)$ is a full spanning fuzzy subgraph of $G = (\sigma, \mu)$ on (V, X) if H is a spanning fuzzy subgraph of G and $\forall\, u, v \in V$ either $\tau(u, v) = 0$ or $\tau(u, v) = \mu(u, v)$.

Theorem 2.8: If $G_1 \times G_2$ is strong fuzzy graph, then at least G_1 or G_2 must be strong.

Proof: On the contrary, assume that both G_1 and G_2 are not strong fuzzy graphs. Then there exists at least one $(u_1, v_1) \in X_1$ and at least one $(u_1, v_1) \in X_1$ such that
(i) $\mu(u_1, v_1) < \sigma_1(u_1) \Lambda \sigma_1(v_1)$ and $\mu_2(u_2, v_2) < \sigma_2(u_2) \Lambda \sigma_2(v_2)$
Without loss of generality we can assume that

(ii) $\mu_2(u_2, v_2) \leq \mu_1(u_1, v_1) < \sigma_1(u_1) \Lambda \sigma_1(v_1) \leq \sigma_1(u_1)$

Now consider $((u_1, u_2)(u_1, v_2)) \in X$. By definition of $G_1 \times G_2$ and inequality (i)

$(\mu_1 \times \mu_2)((u_1, u_2), (u_1, v_2)) = \sigma_1(u_1) \Lambda \mu_2(u_2, v_2) < \sigma_1(u_1) \Lambda \sigma_2(u_2) \Lambda \sigma_2(v_2)$

and

$(\sigma_1 \times \sigma_2)(u_1, u_2), \sigma_1(u_1) \Lambda \sigma_2(u_2) = \sigma_1 \times \sigma_2(u_1, v_2) = \sigma_1(u_1) \Lambda \sigma_2(u_2)$

Thus
$(\sigma_1 \times \sigma_2)(u_1, u_2) \Lambda (\sigma_1 \times \sigma_2)(u_1, v_2) = \sigma_1(u_1) \Lambda \sigma_2(u_2) \Lambda \sigma_2(v_2)$

Hence

$$(\mu_1 \times \mu_2)((u_1, u_2), (u_1, v_2)) < (\sigma_1 \times \sigma_2)(u_1, u_2) \Lambda (\sigma_1 \times \sigma_2)(u_1, v_2)$$

That is, $G_1 \times G_2$ is not strong fuzzy graph, a contradiction. Hence if $G_1 \times G_2$ is strong, then at least G_1 or G_2 must be strong.

Corollary 2.9: If $G_1[G_2]$ is strong then at least G_1 or G_2 is strong.

Definition 2.10: Let (σ, μ) be a fuzzy subgraph of $G = (V, X)$. Denote by X^* the set of all $(u, v) \in X$ for which the strong property fails. That is, $(u, v) \in X^*$ if and only if $\mu(u, v) < \sigma(u) \Lambda \sigma(v)$.

Proposition 2.11: Let (σ_1, μ_1) be a strong fuzzy subgraph of $G_1 = (V_1, X_1)$. Then for any fuzzy graph(σ_2, μ_2) of $G_2 = (V_2, X_2)$, $G_1 \times G_2$ is strong if and only if the following condition is satisfied: for all $u_1 \in V_1$ and $(u_2, v_2) \in X_2^*$, $\sigma_1(u_1) \leq \mu_2(u_2, v_2)$.

Proof: Let $G_1 \times G_2$ be strong. Then for $u_1 \in V_1$, and $(u_2, v_2) \in X_2$, $(\mu_1 \times \mu_2)((u_1, u_2), (u_1, v_2)) = (\sigma_1 \times \sigma_2)(u_1, u_2) \Lambda (\sigma_1 \times \sigma_2)(u_1, v_2) = \sigma_1(u_1) \Lambda \sigma_2(u_2) \Lambda(v_2)$.

By definition, $(\mu_1 \times \mu_2)((u_1, u_2), (u_1, v_2)) = \sigma_1(u_1) \Lambda \mu_2(u_2, v_2)$. Hence
(i) $\sigma_1(u_1) \Lambda \sigma_2(u_2) \Lambda (v_2) = \sigma_1(u_1) \Lambda \mu_2(u_2, v_2)$
If $(u_2, v_2) \in X^*{}_2$, then we have
(ii) $\mu_2(u_2, v_2) < \sigma_2(u_2) \Lambda \sigma_2(v_2)$

From (i) and (ii) it follows that $\sigma_1(u_1) \leq \mu_2(u_2, v_2)$. $(\longleftarrow)$ Conversely, assume $\sigma_1(u_1) \leq \mu_2(u_2, v_2)$ for all $(u_2, v_2) \in X_2^*$ and $u_1 \in V_1$. We want to show that $G_1 \times G_2$ is strong.

Note, $\mu_2(u_2, v_2) < \sigma_2(u_2) \Lambda \sigma_2(v_2)$ and so $\sigma_1(u_1) \Lambda \sigma_2(u_2) \Lambda \sigma_2(v_2) = \sigma_1(u_1) \Lambda \mu_2(u_2 v_2)$ for all $(u, v) \in X_2^*$. For any other $(u_2, v_2) \in X_2$, $\mu_2(u_2, v_2) = \sigma_2(u_2) \Lambda \sigma_2(v_2)$ and so

$$(\mu_1 \times \mu_2)((u_1, u_2)(u_1, v_2)) = \sigma_1(u_1) \Lambda \sigma_2(u_2) \Lambda \sigma_2(v_2)$$

$$= (\sigma_1 \times \sigma_2)(u_1, v_2) \Lambda (\sigma_1 \times \sigma_2)(u_1, v_2)$$

This shows $(\mu_1 \times \mu_2)((u_1, u_2), (u_1, v_2)) = (\sigma_1 \times \sigma_2)(u_1, u_2) \Lambda (\sigma_1 \times \sigma_2)(u_1, v_2)$. If $(u_1, v_1) \in X_1$ and $u_2 \in V_2$ then from the given condition that G_1 is strong it follows that $(\mu_1 \times \mu_2)((u_1, u_2(v_1, u_2)) = \mu_1(u_1, v_1) \Lambda \sigma_2(u_2) = (\sigma_1 \times \sigma_2) \times (u_1, u_2) \Lambda (\sigma_1 \times \sigma_2)(v_1, u_2)$. All this shows that $G_1 \times G_2$ is also strong fuzzy graph. Hence the result.

Theorem 2.12: $G = G^{c^c}$ if and only if G is a strong fuzzy graph.

Theorem 2.13: Let (σ_i, μ_i) be a fuzzy subgraph of $G_i = (V_i, X_i)$ for i = 1, 2. Then the following are true:

(a) $G = G^{c^c}$

(b) $G_i^c = \left(G_i^{c^c}\right)^c$

(c) If $G_1 \subseteq G_2$, then $G_1^{c^c} \subseteq G_2^{c^c}$

Theorem 2.14: G^{c^c} is the smallest strong fuzzy graph that contains G = (V, X). That is, if (σ′, μ′) is a strong fuzzy subgraph of H = (V′, X′) such that $G \subseteq H$, then $G^{c^c} \subseteq H$.

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