International Research Journal of Pure Algebra -3(5), 2013, 201-204
Available online through www.rjpa.info ISSN 2248–9037

Products of $L_2(11)$ by alternating groups

Zishan Liu*

Sichuan University of Science & Engineering, Zigong, Sichuan, 643000, China

(Received on: 08-05-13; Revised & Accepted on: 26-05-13)

ABSTRACT

In this note, we will find the structure of the finite simple groups G with two subgroups A and B such that $G = AB$, where A is a simple group and B is isomorphic to the projective special linear group $L_2(11)$.

Keywords: *Simple group, Factorization, Alternating group, Projective special linear group.*

Mathematics Subject Classification (2010): *20D40.*

1. INTRODUCTION

Let G be a group with subgroups A and B. If $G = AB$, then G is called factorizable group and $G = AB$ is called a factorization of G. Sometimes we say that G is a product of two subgroups A and B. It is an interesting problem to know the groups with proper factorization. Of course not every group has a proper factorization, for example an infinite group with all proper subgroups finite has no proper factorization, $L_2(13)$ and also the Janko simple group J_1 of order 175560 have no proper factorization.

A factorization $G = AB$ is called maximal if both factors A and B are maximal subgroups of G. In [12] all the maximal factorizations of all the finite simple groups and their automorphism groups are found. In [17], all the factorizations of the alternating and symmetric groups are found with both factors simple.

Here we quote some results concerning the alternating groups in a factorization. In [15], factorizable groups where one factor is a non-abelian simple group and the other factor is isomorphic to the alternating group on 5 letters are classified. Also in [4], the structure of finite factorizable group with one factor a simple group and the other factor isomorphic to the symmetric group on 6 letters is determined. In [5], the structure of factorizable groups $G = AB$ where $A \cong A_7$ and $B \cong S_n$ was given. In ([6], the structure of the finite simple factorizable groups $G = AB$ such that A is a non-abelian simple group and $B \cong A_7$, the symmetric group on seven letters is classified. In [13], the structure of products of simple groups with alternating group A_8 of degree eitht is determined. As a development of the topics, we determined the structure of products of an alternating group with $L_2(11)$.

2. PRELIMINARY RESULTS

In this section we obtain results which are needed in the proof of our main theorem. Suppose Ω is a set of cardinality m and G is a k-homogeneous, $1 \le k \le m$, group on Ω.The following Lemma is well-known.

Lemma 2.1: Let G be a k-homogeneous permutation group on a set Ω; $0 \le k \le |\Omega|$. Let H be a k-homogeneous subgroup of G. Then $G = G_\delta H$, where δ is a k-subset of Ω, and G_δ is its stabilizer.

***Corresponding author:** *Zishan Liu**
Sichuan University of Science & Engineering, Zigong, Sichuan, 643000, China

If H is a k-homogeneous subgroup of G, then from [7] we get that the orders of subgroups of $L_2(11)$ are: 1, 2, 3, 4, 5, 6, 10, 11, 12, 55, 60, 660 and the orders of subgroups of $L_2(13)$ are: 1, 2, 3, 4, 7, 12, 13, 14, 26, 39, 1092. Thus the indexes of subgroups of $L_2(11)$ are: 1, 11, 12, 55, 60, 66, 110, 132, 165, 220, 330, 660. It is well-known that $L_2(11)$ has a 2-transitive action on 12 points [2]. Since we need factorizations of the alternating group involving $L_2(11)$, hence using [12], we will prove the following results.

Lemma 2.2: Let A_m denote the alternating group of degree m. If $A_m = AB$ is a non-trivial factorization of A_m where A a non-abelian simple group of A_m and $B \cong L_2(11)$, then one of the following cases occurs:
(a) $A_m = A_{m-1}L_2(11)$, where $m = 11, 12, 55, 60, 66, 110, 132, 165, 220, 330, 660$.
(b) $A_{12} = A_{10}L_2(11)$.
(c) $A_{11} = A_9 L_2(11)$.
(d) $A_{12} = A_9 L_2(11)$.

Proof: It is obvious that m is at least 11. By Theorem D of [12], we have that either $m = 6$, 8 or 10 or one of A or B is k-homogeneous on m letters. Since $m = 6$, 8 or 10, A_m does not involve $L_2(11)$ and so we consider the following cases.

Case (i): $A_{m-k} \trianglelefteq A \le S_{m-k} \times S_k$ for some k with $1 \le k \le 5$, and B is k-homogeneous on m letters.

Since A is assumed to be simple we obtain $A_{m-k} = 1$ or A. If $A_{m-k} = 1$, then $m-k = 1$ or 2, hence $k = m-1$ or $m-2$. But then from $1 \le k \le 5$ we obtain $2 \le m \le 6$ or $3 \le m \le 7$, a contradiction because $m \ge 11$. Therefore $A = A_{m-k}$ and $B \cong A_8$ is k-homogeneous on m letters, $1 \le k \le 5$. If $k = 1$, then by Lemma 2.1, the size of the set Ω on which $L_2(11)$ can act transitively is as stated in the Lemma and all the factorizations in case (a) occur. If $k \ge 2$, then $m = 12$, and so $A_{12} = A_{10}L_2(11)$. This is the Case (b).

Case (ii): $A_{m-k} \trianglelefteq B \le S_{m-k} \times S_k$ for some k with $1 \le k \le 5$, and A is k-homogeneous on m letters.

Since $B \cong L_2(11)$ we obtain $A_{m-k} = 1$ or B and so $m-k =$ 1, 2 or 11. From $1 \le k \le 5$, we have $2 \le m \le 6$, $3 \le m \le 9$ or $12 \le m \le 16$. Therefore, we know that only $m = 12, 13, 14, 15$ or 16 are possible which is correspond to $k = 1, 2, 3, 4, 5$ respectively. We have from Theorem 4.11 and page 197 of [2], and [11], that the possible solutions for (m, k) are (11, 2), (12, 3). Thus $A_{11} = A_9 L_2(11)$ and $A_{12} = A_9 L_2(11)$.

3. MAIN RESULT

To find the structure of the factorizable simple groups $G = AB$ with A simple and $B \cong L_2(11)$, we need to know about the primitive groups of certain degrees which are equal to the indices of subgroups in $L_2(11)$. From [8], we list the primitive permutation groups of degree n less than 1000 as Table 1.

Table - 1: Non-abelian simple primitive groups of degree less than 660.

degree	group
11	A_{11}, M_{11}
12	$A_{12}, L_2(11), M_{11}, M_{12}$
55	$A_{55}, A_{11}, L_2(11)$
60	A_{60}
66	$A_{66}, A_{12}, M_{11}, M_{12}$

110	A_{110}
132	$A_{132}, L_3(8)$
165	A_{165}, A_{11}, M_{11}
220	A_{220}, M_{12}
330	A_{330}
660	A_{660}

Theorem 3.1 Let $G = AB$ is a non-trivial factorization of a simple group G with A a non-abelian simple group and $B \cong L_2(11)$, then one of the following cases occurs:

(a) $A_m = A_{m-1}L_2(11)$, where $m = 11, 12, 55, 60, 66, 110, 132, 165, 220, 330, 660$.

(b) $A_{12} = A_{10}L_2(11)$.

(c) $A_{11} = A_9L_2(11)$.

(d) $A_{12} = A_9L_2(11)$.

(e) $M_{12} = M_{11}L_2(11)$.

Proof: Assume that $G = AB$ is a non-trivial factorization of a simple group G with A a non-abelian simple group and $B \cong L_2(7)$. If M is a maximal subgroup of G containing A, then $G = MB$, hence $\langle |G:M| \, | \, |B:M \cap B| \rangle$. Since $d = |B : B \cap M|$ is equal to the index of a subgroup of A_8, therefore G is primitive permutation group of degree d. We know that $d = 1, 11, 12, 55, 60, 66, 110, 132, 165, 220, 330, 660$. It is easy to see that $d \neq 1$. If G is an alternating group, then from Lemmas 2.1 and 2.2, we have that the cases (a) and (b) is as in the Theorem. Using Table 1, we only consider the following groups: M_{11}, M_{12} and $L_3(8)$.

Let M be a maximal subgroup of G containing A.

If $G = M_{11}$, then $d = |G:M| = 11, 12, 66, 165.$ According to (Conway et al, 1985), we have the foollowing. If $d = 11$ we get $M \cong A_6 \cdot 2$ and so $A = A_6$. Therefore $M_{11} = A_6L_2(11)$. Order consideration, the subgroup of order 30 belongs to both A_6 and $L_2(11)$, a contradiction since A_6 has no subgroup of order 30. If $d = 12,$ then $M \cong L_2(11)$ and so $A = L_2(11)$, which means that $M_{11} = L_2(11)L_2(11) = L_2(11)$, a contradiction. If $d = 66$, then $M \cong S_5$ and so $A \cong A_5$. Hence $M_{11} = A_5L_2(11)$. On the other hand, A_5 is a subgroup of $L_2(11)$ and so $M_{11} = L_2(11)$, a contradiction. If $d = 165$, then $M \cong 2:S_4$ and so $A \cong S_4$, but S_4 is soluble, a contradiction.

If $G = M_{12}$, then $d = 1266220$. According to [1], we have the following. If $d = 12$, then $M \cong M_{11}$ and so $A \cong M_{11}$. Hence $M_{12} = M_{11}L_2(11)$. Since the subgroup of order 55 is both contained in $L_2(11)$ and M_{11}, then this is the case. If $d = 66$, then $M \cong A_6 \cdot 2^2$ and so $A = A_6$. Order consideration rules out the case. If $d = 220$, then $M \cong 3^2 : 2S_4$. We rule out this case.

If $G = L_3(8)$, then there is no subgroup of index 132 and so we rule out this case.

This completes the proof of the Theorem.

ACKNOWLEDGMENTS

The author is very grateful for the helpful suggestions of the referee. This work was supported by Foundation of Development Research Center for Liquor Industry of Sichuan (CJY12-37)

REFERENCES

[1] M. Blaum, Factorizations of the simple groups PSL (3; q) and PSL (3; q^2), Arch. Math, 40(1983), 8-13.

[2] P. J. Cameron, Permutation groups, Cambridge University Press, Cambridge, 1999.

[3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker. and R. A. Wilson, Atlas of Finite Groups. Clarendon Press, Oxford, 1985

[4] M. R. Darafsheh, G. R. Rezaeezadeh and G. L. Walls, Groups which are the product of S_6 and a simple group, Alg. Coll, 10(2)(2003), 195-204.

[5] M. R. Darafsheh, Product of the symmetric group with the alternating group on seven letters. Quasigroups Related Systems, 9(2002), 33-44.

[6] M. R. Darafsheh, Factorization of simple groups involving the alternating-group, Quasigroups Related Systems, 13(2005), 203-211.

[7] L. E. Dickson, Linear groups with an exposion of the Galois field theory. Leipzig, 1901.

[8] J. D. Dixon and B. Mortimer, The primitive permutation groups of degree less than 1000, Math. Proc. Camb. Phil. Soc, 213(1988), 213-238.

[9] T. R. Gentchev, Factorizations of the sporadic simple groups. Arch. Math, 47(1986), 97-102.

[10] N. Ito, On the factorizations of the linear fractional groups LF (2, p^n). Acta Sci. Math., 15(1953), 79-84.

[11] W. M. Kantor, k-homogeneous groups. Math. Z, 124(1972), 261-265.

[12] M. W. Libeck, C. E. Praeger and J. Saxl, The maximal factorizations of the finite simple groups and their automorphism groups, Memoirs AMS, 86(1990).

[13] S. Liu, Products of simple group involving the alternating group A_8 , manuscript.

[14]B. Mwene, On some subgroups of PSL (4; q), q odd. Geome. Dedi., 12(1982), 189-199.

[15] W. Scott, Products of A_5 and a finite simple group, J. Algebra, 37(1975), 165-171.

[16] C. M. Roney-Dougal, The primitive permutation groups of degree less than 2500, J. Algebra, 292(2005), 154-183.

[17] G. L. Walls, Non-simple groups which are the product of simple groups, Archiv der Mathematik, 53(1989), 209-216.

[18] R. Wilson, et al. ATLAS of Finite Group Representations. http://brauer.maths.qmul.ac.uk/Atlas/v3.

[19] A. V. Zavarnitsine, Finite simple groups with narrow prime spectrum. Sibe. Elec. Math. Rep. 6(2009), 1-12.

Source of support: Foundation of Development Research Center for Liquor Industry of Sichuan (CJY12-37), Conflict of interest: None Declared