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PROPERTY OF HADAMARD PRODUCT OF MATRICES OVER SKEW FIELD

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ABSTRACT

In this paper, we give $Hadamard$ *product over skew field and relevant property by extending the property of* $Hadamard$ *product of two matrices on complex domain to skew field. Especially, this paper using the method of block matrices gets a few* $Hadamard$ *products' preference ordering Inequalities about Positive Semi-definite matrices* $M-P$ *inverse.*

Keywords: $Hadamard$ *product; positive semi-definite self-conjugate matrix; skew field.*

1. INTRODUCTION

$Hadamard$ product and its property over skew field have been studied in some literatures. But we have not detailed reported for property of the $Hadamard$ product of matrices over skew field so far. In this paper, we further study the related property and inequalities of the $Hadamard$ product over skew field by extending the $Hadamard$ product of two matrices on general complex domain to skew field. And based on the method of block matrices, we can obtain several $Hadamard$ product inequalities about positive semi-definite matrices. And there are the similar conclusions on the skew field.

2. PREPARATION KNOWLEDGE

In this paper, K denotes a skew field, $K^{m\times n}$ represents the set of $m\times n$ order matrix, $M_n(K)$ represents the set of $n\times n$ order matrix over K, $GL_n(K)$ means the all of the inverse matrix of n order , A^* means the conjugate transpose matrix of A. $H(n,*)$ denotes the set of self-conjugate matrix of order n. $Z(K)$ is the centre of K ,let $F=Z(K)$, so $M_n(F)$ denotes the set of communicate matrix of order n over K. Moreover, If

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \text{ then } diagA=\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

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Definition 1[2]**:** Let $A=(a_{ij})\in K^{m\times n}$, $B=(b_{ij})\in K^{m\times n}$, definite the $Hadamard$ product of A and B as follows:

$$A\circ B=[a_{ij}b_{ij}]=\begin{pmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1n}b_{1n} \\ a_{21}b_{21} & a_{22}b_{22} & \cdots & a_{2n}b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}b_{m1} & a_{m2}b_{m2} & \cdots & a_{mn}b_{mn} \end{pmatrix}.$$

Definition 2[1]**:** Let R is a non-commutative principal ideal domian with involutorial anti-automorphism σ, $A\in R^{m\times n}$, if there exits $X\in R^{n\times m}$ such that:

$$AXA=A,$$

$$XAX=X,$$

$$(AX)^*=AX,$$

$$(XA)^*=XA.$$

Then, we say X is a $Moore-Penrose$ inverse of A, denoted A^+.

Definition 3[1]**:** Let $A\in H(n,*)$, $H(n,*)$ denotes the set of self-conjugate matrix of order n.If $\alpha^*A\alpha\ge 0$ holds for any n dimension and non-zero column vector, then we say A is a positive semi-definite self-conjugate matrix of order n.we rewrite it： $A\ge 0$

Definition 4[1]**:** Let $A\in M_n(R)$, if there is a $B\in M_n(R)$, such that $AB=BA=I_n$,then we call A is a inverse matrix of n order. The inverse of A is denoted A^{-1}.

The following lemmas are the basis of the conclusions we talking:

Lemma 1[3][4]**:** $M=\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\in M_n(K)$ and $M\ge 0$, where $A,B,C\in M_n(K)$, then $C-B^*A^+B\ge 0$

Lemma 2[4]**:** For any $A,B\in K^{m\times n}$, we have $A\circ B=M^T(A\otimes B)M$, where M is the $n^2\times n$ selection matrix， N is the $m^2\times m$ selection matrix. (M and N are the matrices which elements only are 1 and 0 ,and $M^TM=I_n$, $N^TN=I_m$), $A\otimes B$ is the $Kronecker$ product of A and B.This lemma is the connection of the $Kronecker$ and $Hadamard$ product.

Lemma 3: Let $A,B\in M_n(K)$, and $A\ge 0$, $B\ge 0$,then $M_i\ge 0 (i=1,2,3,4)$, where $M_1 = \begin{bmatrix} BB^+ & B \\ B & B^2 \end{bmatrix}$, $M_2 = \begin{bmatrix} A^+ & AA^+ \\ A^+A & A \end{bmatrix}$,

$M_3 = \begin{bmatrix} B^2 & B \\ B & BB^+ \end{bmatrix}$, $M_4 = \begin{bmatrix} A & AA^+ \\ AA^+ & A^+ \end{bmatrix}$.

Proof: $\begin{bmatrix} I_n & 0 \\ -B & I_n \end{bmatrix} M_1 \begin{bmatrix} I_n & -B \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} BB^+ & 0 \\ 0 & 0 \end{bmatrix} \geq 0$,

$\begin{bmatrix} I_n & 0 \\ -A & I_n \end{bmatrix} M_2 \begin{bmatrix} I_n & -A \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} A^+ & 0 \\ 0 & 0 \end{bmatrix} \geq 0$,

Similarly, $M_3 \geq 0$， $M_4 \geq 0$.

Lemma 4: Let $A, B \in M_n(K)$， moreover $A \geq 0$, $B \geq 0$, if $\begin{pmatrix} A & B \\ B & A \end{pmatrix} \geq 0$,then $A \geq B$.

Proof: $2(A-B) = (A-B, B-A) \begin{bmatrix} I_n \\ -I_n \end{bmatrix} = (-I_n, I_n) \begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} I_n \\ -I_n \end{bmatrix} \geq 0$,

So $A - B \geq 0$，即 $A \geq B$.

3. PROPERTY AND THEOREM

Hadamard meet the following basic properties:

(1) $A = (a_{ij}) \in K^{m\times n}$, null matrix $\mathbf{0} \in M_n(K)$,则 $A \circ 0 = 0 \circ A = 0$.

(2) $(A \circ B)^T = A^T \circ B^T$.

(3) $c(A \circ B) = (cA) \circ B$, c is a real number.

(4) The *Hadamard* product of $A = (a_{ij}) \in K^{m\times m}$ and I_m is a $m \times m$ diagonal matrix namely：

$A \circ I_m = I_m \circ A = diag(A) = diag(a_{11}, a_{22} \cdots a_{mm})$.

(5) $A \circ (B \circ C) = (A \circ B) \circ C = A \circ B \circ C$

$(A \pm B) \circ C = A \circ C \pm B \circ C$

$(A + B) \circ (C + D) = A \circ C + A \circ D + B \circ C + B \circ D$.

(6) If $A = (a_{ij}) \in K^{m\times m}, C = (c_{ij}) \in K^{m\times m}$

$B = (b_{ij}) \in K^{n\times n}, D = (d_{ij}) \in K^{n\times n}$

Then $(A \oplus B) \circ (C \oplus D) = (A \circ C) \oplus (B \circ D)$.

Proof: $A\oplus B=\begin{bmatrix} A & 0_{m\times n} \\ 0_{n\times m} & B \end{bmatrix}, C\oplus D=\begin{bmatrix} C & 0_{m\times n} \\ 0_{n\times m} & D \end{bmatrix}$

$$(A\oplus B)\circ(C\oplus D)=\begin{bmatrix} A\circ C & 0_{m\times n} \\ 0_{n\times m} & B\circ D \end{bmatrix}$$

$$(A\circ C)\oplus(B\circ D)=\begin{bmatrix} A\circ C & 0_{m\times n} \\ 0_{n\times m} & B\circ D \end{bmatrix}$$

This proposition is proved.

(7) If A,B,D are the square matrix of $M_n(K)$, D is a diagonal matrix, so $(DA)\circ(BD)=D(A\circ B)D$.

(8) $tr(A\circ B)\neq tr(A)tr(B)$.

(9) Because the skew field (decision ring) is non-commutative, $A\circ B\neq B\circ A$.

For example: $A=\begin{pmatrix} 1 & i \\ j & k \end{pmatrix}$, $B=\begin{pmatrix} j & k \\ i & 1 \end{pmatrix}$,

Then $A\circ B=\begin{pmatrix} j & ik \\ ji & k \end{pmatrix}=\begin{pmatrix} j & -j \\ -k & k \end{pmatrix}$, $B\circ A=\begin{pmatrix} j & ki \\ ij & k \end{pmatrix}=\begin{pmatrix} j & j \\ k & k \end{pmatrix}$,

So $A\circ B\neq B\circ A$.

(10) $A,B\in K^{m\times n}$, If A,B are positive semi-definite, then its *Hadamard* product $A\circ B$ is also positive semi-definite.

Theorem 1: If $A,B,C\in K^{m\times n}$, then $tr\left[A^T(B\circ C)\right]=tr\left[\left(A^T\circ B^T\right)C\right]$.

Proof: Noted the diagonal elements of $A^T(B\circ C)$ and $\left(A^T\circ B^T\right)C$ are common, that is:

$$\left[A^T(B\circ C)\right]_{ii}=\sum_{k=1}^{n}a_{ki}b_{ki}c_{ki}=\left[\left(A^T\circ B^T\right)C\right]_{ii}.$$

Theorem 2: If $A,B\in K^{m\times n}$, then $rank(A\circ B)\leq(rankA)(rankB)$.

Proof: Any matrix with rank r can be written as combination of r matrices with rank 1 . Every matrix with rank 1 is the outer product of two vectors. If $rankA=r_1, rankB=r_2$, then $A=\sum_{i=1}^{r_1}x_iy_i^T$, $B=\sum_{j=1}^{r_2}u_jv_j^T$, where x_i,u_j and y_i,v_j are the linear independent column vector over

K, $i=1,2,\cdots r_1$, $j=1,2,\cdots r_2$. Therefore, $A\circ B=\sum_{i=1}^{r_1}\sum_{j=1}^{r_2}\left(x_i\circ u_j\right)\left(y_i\circ v_j\right)^T$. Then

$$rank(A\circ B)= rank\left[\sum_{i=1}^{r_1}\sum_{j=1}^{r_2}(x_i\circ u_j)(y_i\circ v_j)^T\right]\le \sum_{i=1}^{r_1}\sum_{j=1}^{r_2} rank\left[(x_i\circ u_j)(y_i\circ v_j)^T\right]$$

$$\le \sum_{i=1}^{r_1}\sum_{j=1}^{r_2} 1 = r_1 r_2 = (rankA)(rankB).$$

This suggests that $A\circ B$ could be written as combination of at most $r_1 r_2$ matrices with 1 rank.

So, $rank(A\circ B)\le (rankA)(rankB)$.

Another method: Because $A\circ B = M^T(A\otimes B)N$, according to the property of matrix of rank:

$$rank(A\circ B)= rank[M^T(A\otimes B)N]\le rank(A\otimes B).$$

Next we should show $rank(A\otimes B)=(rankA)(rankB)$

Let $r(A)=s$, then there exists $P\in GL_m(K), Q\in GL_n(K)$ such that $PAQ=\begin{pmatrix} I_s & 0\\ 0 & 0\end{pmatrix}$. For $B\in K^{m\times n}$, let

$r(B)=t$ there exists $S\in GL_m(K), T\in GL_n(K)$ such that

$SBT=\begin{pmatrix} I_t & 0\\ 0 & 0\end{pmatrix}$.because $(P\otimes S)(A\otimes B)(P\otimes S)(A\otimes B)=(PAQ)\otimes(SBT)=\begin{pmatrix} I_s & 0\\ 0 & 0\end{pmatrix}\otimes\begin{pmatrix} I_t & 0\\ 0 & 0\end{pmatrix}=$

$\begin{pmatrix} I_{st} & 0\\ 0 & 0\end{pmatrix}$. Because $P\otimes S$, $Q\otimes T$ are invertible matrix, $r(A\otimes B)=r\begin{pmatrix} I_{st} & 0\\ 0 & 0\end{pmatrix}=st=r(A)\,r(B)$.

So, $rank(A\circ B)\le (rankA)(rankB)$.

Theorem 3: Let $A,B\in H(n,*)$ and $A\ge 0$, $B\ge 0$,then $A^+\circ B^+\ge [(A^+A)\circ(B^+B)](A\circ B)^+[(AA^+)\circ(BB^+)]$.

Proof: By lemma3 and theorem 1：

$$\begin{pmatrix} A\circ B & (AA^+)\circ(BB^+)\\ (A^+A)\circ(B^+B) & A^+\circ B^+\end{pmatrix}=\begin{pmatrix} A & AA^+\\ A^+A & A^+\end{pmatrix}\circ\begin{pmatrix} B & BB^+\\ B^+B & B^+\end{pmatrix}\ge 0.$$

According to the definition of $Moore-Penrose$ inverse:

$$[(AA^+)\circ(BB^+)]^* = (AA^+)^*\circ(BB^+)^* = (A^+A)\circ(B^+B)$$

Remark: $A\circ B = C,\ (AA^+)\circ(BB^+)=D,\ A^+\circ B^+ = G$

Then $\begin{pmatrix} A\circ B & (AA^+)\circ(BB^+)\\ (A^+A)\circ(B^+B) & A^+\circ B^+\end{pmatrix}=\begin{pmatrix} C & D\\ D^* & G\end{pmatrix}\ge 0$

According to lemma 2, $G-D^*C^+D\ge 0$,

so , $A^+\circ B^+ - [(A^+A)\circ(B^+B)](A\circ B^+)[(AA^+)\circ(BB^+)]\ge 0$.

Corollary 1: Assume that $A, B \in H(n,*)$ and $A \geq 0$, $B > 0$, then

$$A^{+} \circ B^{-} \geq (diagA^{+}A)(A \circ B)^{+}(diagAA^{+})$$

Proof: According to theorem 1, let $B^{+} = B^{-}$, then $B^{-}B = BB^{-} = I$, on the basis of property (4), We can come to this conclusion easily.

Corollary 2: Assume that $A, B \in H(n,*)$ and $A > 0$, $B > 0$, then $A^{-} \circ B^{-} \geq (A \circ B)^{-}$.

Proof: Let $A^{+} = A^{-}$, $B^{+} = B^{-}$, the result can be obtained.

Theorem 4: Assume that $A, B \in H(n,*)$ and $A \geq 0$, $B \geq 0$ then $A^{2} \circ B^{2} \geq (A \circ B)[(AA^{+}) \circ (BB^{+})]^{+}(A \circ B)$

Proof: Combing the lemma 3 and theorem 1

$$\begin{pmatrix} (AA^{+}) \circ (BB^{+}) & A \circ B \\ A \circ B & A^{2} \circ B^{2} \end{pmatrix} = \begin{pmatrix} AA^{+} & A \\ A & A^{2} \end{pmatrix} \circ \begin{pmatrix} BB^{+} & B \\ B & B^{2} \end{pmatrix} \geq 0$$

By lemma 2, $A^{2} \circ B^{2} - (A \circ B)[(AA^{+}) \circ (BB^{+})]^{+}(A \circ B) \geq 0$.

Corollary 3: Assume that $A, B \in H(n,*)$ and, then $A^{2} \circ B^{2} \geq (A \circ B)^{2}$.

Theorem 5: A is a self-conjugate matrix on $M_{n}(F)$, and $A \geq 0$, then

$$(AA^{+}) \circ A^{2} \geq A \circ A.$$

Proof: According to lemma 3 and theorem 1:

$$\begin{pmatrix} A^{2} \circ (AA^{+}) & A \circ A \\ A \circ A & A^{2} \circ (AA^{+}) \end{pmatrix} = \begin{pmatrix} (AA^{+}) \circ A^{2} & A \circ A \\ A \circ A & A^{2} \circ (AA^{+}) \end{pmatrix} = \begin{pmatrix} AA^{+} & A \\ A & A^{2} \end{pmatrix} \circ \begin{pmatrix} A^{2} & A \\ A & AA^{+} \end{pmatrix} \geq 0$$

By lemma 4, $(AA^{+}) \circ A^{2} \geq A \circ A$.

Corollary 4: A is a self-conjugate matrix on $M_{n}(F)$, then $diagA^{2} \geq A \circ A$

Corollary 5: By corollary 4, if all the diagonal elements of A^{2} are 1, then $A \circ A \leq I_{n}$

Theorem 6: A is a self-conjugate matrix on $M_{n}(F)$, and $A \geq 0$, then $A \circ A^{+} \geq (A^{+}A) \circ (AA^{+})$

Proof: Combing the lemma 3 and theorem 1:

$$\begin{pmatrix} A \circ A^{+} & (AA^{+}) \circ (A^{+}A) \\ (A^{+}A) \circ (AA^{+}) & A \circ A^{+} \end{pmatrix} = \begin{pmatrix} A \circ A^{+} & (AA^{+}) \circ (AA^{+}) \\ (A^{+}A) \circ (A^{+}A) & A^{+} \circ A \end{pmatrix} = \begin{pmatrix} A & AA^{+} \\ A^{+}A & A^{+} \end{pmatrix} \circ \begin{pmatrix} A^{+} & AA^{+} \\ A^{+}A & A \end{pmatrix} \geq 0$$

by lemma 4, $(AA^{+}) \circ A^{2} \geq A \circ A$

Theorem 7: Assume that $A, B, M \in K^{m \times n}$, $A \geq 0$, $B \geq 0$, $M \geq 0$, for any real number a, b, we have:

$$(a^{2} + b^{2})(A^{*}A \circ B^{*}B) + 2ab(A^{*}B \circ B^{*}A) \geq (a+b)^{2}(A^{*}M \circ B^{*}M)(M^{*}M \circ MM^{*})^{+}(M^{*}A \circ M^{*}B)$$

Proof: Remark: $L = \begin{pmatrix} L_1 & L_2 \end{pmatrix}$, where $L_1 = M \otimes M$, $L_2 = aA \otimes B + bB \otimes A$.

$$S = L^* L = \begin{pmatrix} L_1^* \\ L_2^* \end{pmatrix} \begin{pmatrix} L_1 & L_2 \end{pmatrix} = \begin{pmatrix} L_1^* L_1 & L_1^* L_2 \\ L_2^* L_1 & L_2^* L_2 \end{pmatrix} \geq 0$$

Though calculation:

$$L_1^* L_1 = (M^* M \otimes M^* M)$$

$$L_1^* L_2 = a\ M^*\ A \otimes\ M^* B + bM^* B \otimes M^* A$$

$$L_2^* L_1 = aA^* M \otimes B^* M + bB^* M \otimes A^* M$$

$$L_2^* L_2 = a^2 (A^* A \otimes B^* B) + b^2 (B^* B \otimes A^* A) + ab[(A^* B \otimes B^* A) + (B^* A \otimes A^* B)]$$

Combing the lemma 3 and theorem 1, the result can be held up.

Corollary 6: If $A, B \in K^{m\times n}$, and $k \in [-1,1]$,then $AA^* \circ BB^* + kAB^* \circ BA^* \geq (1+k)(A \circ B)(A \circ B)^*$

Proof: By theorem8, let $a^2 + b^2 = 1,\ 2ab = k,\ M = I$, we can see the results easily.

Corollary 7: If $A, B \in K^{m\times n}$, then $A^* A \circ B^* B \geq (A^* \circ B^*)(A \circ B)$

Proof: By theorem8, let $a = 1, b = 0, M = I$, the results can be held.

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