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HOMOMORPHISM AND ISOMORPHISM THEOREMS OF SEMI MODULES OVER A BOOLEAN LIKE SEMI RING

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ABSTRACT

In this paper, the concepts semi module homomorphism and isomorphism over a Boolean like semi ring R areintroducedand also study some of its properties. Further fundamental theorem of homomorphism of semi modules and also isomorphism theorems are proved.

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Key Words: *Boolean like semi ring, Quotient Boolean like semi ring, Semi module, Quotient semi Module andsemi module homomorphism and isomorphism.*

INTRODUCTION

The concept of Semi Modules over a Boolean like Semi Ringis due to Murthy B.V.N [3]. This paper is further study on semi module over a Boolean like semi ring has been made.The present author has introduced the concept of semi module homomorphism and isomorphism over a Boolean like semi ring and obtain isomorphism theorems. The present paper is divided into2 sections. In section 1, the preliminary concepts and results regarding of Semi Modules over a Boolean like Semi Ring, are given. In section 2, obtain the formal properties of semi module homomorphism such as the homomorphic image of a semi module is semi module and further Ker of homomorphism is defined and obtain it is an R-ideal .Also obtain the fundamental theorem ofsemi module homomorphism and isomorphism theorems for semi modules.

Finally end this section with the isomorphism theorems (see 2.8 and 2.9)

1. PRELIMINARIES

We recall certain definitions and results concerning Semi Modules over a Boolean like Semi Ringfrom [2, 3]

Definition 1.1: A non-empty set R together with two binary operations + and . satisfying the following conditions is called a Boolean like semi ring

1. (R, +) is an abelian group
2. (R, .) is a semi group
3. a. (b+c) = a.b +a.c for all a, b, c $\in$ R
4. a + a = 0 for all a $\in$ R
5. ab (a+b+ab) = ab for all a, b $\in$ R.

Let R be a Boolean like semi ring. Then

Lemma 1.2: For a $\in$ R, a.0 =0

Lemma 1.3: For a $\in$ R, $a^4 = a^2$ (weak idempotent law)

Remark 1.4: If R is a Boolean like semi ring then, $a^n = a$ or a^2 or a^3 for any integer n > 0

Definition 1.5: A Boolean like semi ring R is said to be weak commutative if abc = acb, for all a, b, c$\in$ R.

Lemma 1.6: If R is a Boolean like semi ring with weak commutative then 0.a = 0, for all a$\in$R

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Lemma 1.7: Let R be Boolean like semi ring then for any a, b $\in$ R and for any integers m, n > 0
1. $a^m a^n = a^{m+n}$
2. $(a^m)^n = a^{mn}$
3. $(ab)^n = a^n b^n$
if R is weak commutative.

Definition 1.8: A non empty subset I of R is said to be an ideal if
1. (I,+) is a sub group of (R,+) , i.e. for a, b $\in$ I $\Rightarrow$ a + b $\in$ I
2. ra $\in$ I, for all a $\in$ I, r $\in$ R, i.e. RI $\subseteq$ I
3. (r+a) s + rs $\in$ I. for all r, s $\in$ R, a $\in$ I

Remark 1.9: If I satisfies 1 and 2, I is called left ideal and If I satisfies 1 and 3, I is called right ideal of R.

Remark 1.10: If R is weak commutative Boolean like semi ring then ar $\in$ I for all a $\in$ I and r $\in$ R.

Definition1.11: Let R be a Boolean like semi ring and (M, +) be an abelian group then M is called a semi module over R if there is a mapping . : M x R $\rightarrow$ M (the image of (m, r) under th mapping is denoted by mr) such that

m(r+s) = mr + ms and m(rs) = (mr)s, for all m$\in$ M, r, s$\in$ R.

Definition 1.12: Let H be a sub group of M such that for all r $\in$R, for all h $\in$ H, we have that hr$\in$ H then H is called R – sub module of M, we denote $H <_R M$

Definition1.13: If R is weak commutative and M is module over R, 0$\in$M, 0r = 0, for all r$\in$ R.

Theorem 1.14: If M is semi module over R then m0 = 0 for all m $\in$M.

Definition1.15: A sub group P of a module M is called R-ideal of M if for all r $\in$ R m $\in$M and n $\in$P, we have

(m + n)r – mr$\in$ P.

Remark1.16: If M = R then R – ideals of M becomes right ideals of R and the R – sub modules of M are the right R – sub groups of R..

Theorem1.17: If R is weak commutative then every R- ideal of M is an R- sub module of M.

Proposition1.18: Let I be an ideal of R then R/I is semi module over R with scalar multiplication defined by:
(s+I)r = sr +I for all r, s $\in$R.

Definition1.19: Let M be a semi module over R and P be an R- ideal of M. Then the quotient group
M/P = {m + P/m$\in$M} is semi module over R (called the quotient semi module over R) with scalar multiplication defined by

(m + P) r = mr + P, for all r $\in$R, m$\in$M.

Theorem 1.20: Let M be an R- semimodule, H an R – sub module of M and K an R-sub module (R-ideal) of M then
H $\cap$K is an R- sub module (R-ideal) of M

Theorem1.21: Let M be an R- semi module, H an R - sub module of M and K an R-ideal of M then H + K is an R- sub module of M.

2. SEMI MODULE HOMOMORPHISM AND ISOMORPHISM

Definition2.1: Let M, N be two semi modules over R and f: M$\rightarrow$ N is called R- semi module homomorphism if

$f(m_1 + m_2) = f(m_1) + f(m_2)$ and f(mr) = f(m)r, for all $m_1, m_2,$ m$\in$M, r $\in$R.

Theorem 2.2: Let M, N be two semi modules over R and f: M $\rightarrow$N be R-semi module homomorphism then
(i) f (0) = 0
(ii) f (-m) =-f(m)
(iii) $f(m_1 - m_2) = f(m_1) - f(m_2)$
(iv) f(M) = {f(m) / m$\in$ M} is R-sub module of N.
Proof is routine verification.

Theorem 2.3: Let M, N be two semi modules over R and f: M $\rightarrow$N be R-semi module homomorphism then Ker f = {m $\in$ M / f(m) = 0,0 is the identity element of the additive group of N }, is called the kernel of homomorphism f, which is an R- ideal of M and also R- submodule of M.

Proof: From Theorem 2.2, Ker f is non-empty, as 0 $\in$ Ker f, Let $m_1, m_2 \in$ Ker f, r $\in$R then $f(m_1) = 0$, $f(m_2) = 0$
$f(m_1 - m_2) = f(m_1) - f(m_2) = 0-0 = 0$

$m_1 - m_2 \in$Ker f. Hence Ker f is sub group of M.

Let m $\in$M, r $\in$R, x$\in$ Ker f $\Rightarrow$ f(x) = 0, we show that (m+x)r –mr $\in$Ker f

Now f [(m+x)r –mr] = f [(m+x)r] –f(mr) = f(m+x) r –f(m) r = [f(m)+f(x)]r –f(m) r

= [f(m)+0]r –f(m) r = f(m) r –f(m) r = 0

Thus Ker f is an R- ideal of M.

We now show that xr $\in$ Ker f

x$\in$ Ker f $\Rightarrow$ f(x) = 0. Consider f(xr) = f(x)r = 0r = 0, Ker f is R- sub module of M.

Definition 2.4: Let M, N be two semi modules over R and f: M $\rightarrow$ N is called R- semi module isomorphism if
(i) f is one-one and onto
(ii) $f(m_1 + m_2) = f(m_1) + f(m_2)$
(iii) f(mr) = f(m)r, for all m_1, m_2, m$\in$M, r $\in$R.

If f: M $\rightarrow$ N is R- semi module isomorphism then M is isomorphic to N, It is denoted by $M \approx_R N$.

Theorem 2.5: Let M, N be two semi modules over R and f: M $\rightarrow$N be R-semi module homomorphism then f is one-one if and only if Ker f = {0}.

Proof: Suppose f is one-one. Let m$\in$ Ker f $\Rightarrow$ f(m) = 0

f(m) = 0 = f(0) (f is one-one) $\Rightarrow$ m = 0

Hence Ker f = {0}.

Conversely, $m_1, m_2 \in$M such that $f(m_1) = f(m_2) \Rightarrow f(m_1) - f(m_2) = 0 \Rightarrow f(m_1-m_2) = 0$,

$\Rightarrow m_1 - m_2 = 0 \Rightarrow m_1 = m_2$

Hence f is one – one.

Theorem2.6: (Fundamental theorem of semi module homomorphism). Let M, N be two semi modules over R and f: M $\rightarrow$N be R-semi module homomorphism with Ker f = K then $\frac{M}{K} \approx_R$Im(f).

Proof: Define g: $\frac{M}{K} \rightarrow$ Im(f) by g(m+K) = f(m), for all m+K$\in\frac{M}{K}$

For $m_1+K, m_2+K \in \frac{M}{K}$ such that $m_1+K = m_2+K \Leftrightarrow m_1 - m_2 \in K$ = Ker f $\Leftrightarrow f(m_1 - m_2) = 0 \Leftrightarrow m_1 - m_2 \in$Ker f = K$\Leftrightarrow m_1+K = m_2+K$. Hence g is well defined and one-one.

g is onto: Let n $\in$Im(f) then there exists m $\in$M such that f(m) = n

Now m $\in$M then m+K$\in\frac{M}{K}$ and g(m+K) = f(m) = n

g is homomorphism: For $m_1+K, m_2+K \in \frac{M}{K}$, r $\in$R

$g((m_1+K) + (m_2+K)) = g((m_1+m_2)+K) = f(m_1+m_2)$
$= f(m_1) + f(m_2) = g((m_1+K) + g(m_2+K))$

g[(m+K)r] = g (mr+K) = f(mr) = f(m)r = [g(m+K)]r

Thus g :$\frac{M}{K}$→ Im(f) is an isomorphism and hence $\frac{M}{K} \approx_R$ Im(f).

Theorem 2.7: If M is R- semi module and P be R-ideal of M then f: M → M/P by f(m) = m+P, for all m∈M, is an onto homomorphism with Ker f = P.

Proof: For m, m_1, m_2∈ M, r∈R,

f(m_1 + m_2) =(m_1 + m_2)+P = (m_1 +P)+(m_2+P) = f(m_1) +f(m_2),

f(mr) = mr+P = (m+P)r = f(m)r, f is homomorphism.

Clearly f is onto. Now Ker f = {m∈M/ f(m) = 0+P, zero element in M/P}
= {m∈M/ m+P = 0+P} = {m∈M/m∈P} = P.

Theorem 2.8: Let M be R-semi module. Let K be R- ideal of M and H be R- sub module of M then $\frac{K+H}{K} \approx_R \frac{H}{K \cap H}$.

Proof: Define g: H $\to \frac{K+H}{K}$ by g(h) = h+K, for all h ∈ H.

For h_1, h_2∈ H such that h_1= $h_2$thenh$_1$+K = h_2+K ⇒ g(h_1) = g(h_2), g is well defined.

g is onto: Let x+K∈ $\frac{K+H}{K}$, then x∈ K+H , let x = k+h , k∈K, h∈H

If k∈K then k+K = 0+K. Now g(h) = h+K = h+(k+K) = (h+k)+K = x+K

g is homomorphism: For h, h_1, h_2∈ H, r∈R, g(h_1+ h_2) = (h_1+ h_2) + K = (h_1+ K) + (h_2+ K) = g(h_1) + g(h_2)

g(hr) = hr + K = (h+K)r = g(h) r.

Ker g = $\left\{ \mathrm{h} \in \mathrm{H} / \mathrm{g}(\mathrm{h}) = 0 + \mathrm{K},\ \text{zero element in} \frac{K+H}{K} \right\} = \{ h \in H / h + K = K \}$
= {h∈H /h∈K} = K∩H

Hence by fundamental theorem of semi module homomorphism, $\frac{H}{\ker g} \approx_R \frac{K+H}{K}$

i.e., $\frac{H}{K \cap H} \approx_R \frac{K+H}{K}$

Theorem 2.9: Let M be R-semi module. Let H and K be R- ideals of M such that H ⊆K then $\frac{M/H}{K/H} \approx_R \frac{M}{K}$

Proof: Define g: $\frac{M}{H} \to \frac{M}{K}$ by g(m+H) = m+K , for all m∈M.

For m_1+H, m_2 +H ∈ $\frac{M}{H}$ such that m_1+ H = m_2 + H ⇒ m_1- m_2∈H⊆K

⇒ m_1+K = m_2+K ⇒g(m_1+H)= g(m_2+H) , g is well defined. Clearly g is onto.

g is homomorphism: For m + H, m_1+H, m_2 +H ∈ $\frac{M}{H}$, r∈R

Now $g[(m_1+H)+(m_2+H)] = g((m_1+m_2)+H) = ((m_1+m_2)+K) = (m_1+K)+(m_2+K)$

$$= g[(m_1+H)]+g[(m_2+H)]$$

$g[(m+H)r] = g[(mr+H)] = (mr+K) = (m+K)r = [g(m+H)]r$

$$\text{Ker } g = \left\{ m+H / g(m+H) = 0+K, \text{ zero element in } \frac{M}{K} \right\}$$

$$= \{ m+H = 0+K \}$$

$$= \{ m+H / m \in K \} = \frac{K}{H}$$

Hence by fundamental theorem of semi module homomorphism,

$$\frac{M/H}{K/H} \approx_R \frac{M}{K}$$

REFERENCES

[1] Mr. S. Juglal: Prime near ring modules and their links with the generalized group near -ring

[2] Venkateswarlu. K and Murthy B.V.N and Amaranth N: Boolean like semi rings, Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 13, 619 – 635

[3] Murthy B. V. N: A Structure of Semi module over a Boolean like Semi ring, Int. J. Mathematical Archive-3(8), 2012, 2850-2857.

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