(τ₁, τ₂)* - Q* HOMEOMORPHISM IN BITOPOLOGICAL SPACES

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ABSTRACT

The aim of this paper is to introduce and study the new type of homeomorphism, namely (τ₁, τ₂)* - Q* homeomorphism and τ₁τ₂ - Q* homeomorphism in bitopological spaces. Also we define (τ₁, τ₂)* - irreducible spaces and τ₁τ₂ - irreducible spaces. Here researchers proved that the set of all (τ₁, τ₂)* - Q* homeomorphism forms a group.

Keywords: (τ₁, τ₂)* - Q* homeomorphism, τ₁τ₂ - Q* homeomorphism, (τ₁, τ₂)* - irreducible spaces and τ₁τ₂ - irreducible spaces.

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1. INTRODUCTION

The notion of homeomorphism plays a dominant role in topology and so many authors introduced various types of homeomorphisms in topological spaces. In 1995, Maki, Devi and Balachandran[ 3] introduced the concepts of semi-generalized homeomorphisms and generalized semi-homeomorphisms and studied some semi topological properties. Devi and Balachandran introduced a generalization of α-homeomorphism in 2001.

Recently, P. Padma and S. Udayakumar [8] introduced and studied the concept of (τ₁, τ₂)* - Q* continuous maps in bitopological spaces.

The purpose of this paper is to introduce the concepts of homeomorphisms by using (τ₁, τ₂)* - Q* open sets. In this paper, we introduce the concepts τ₁τ₂ - Q* homeomorphism, and (τ₁, τ₂)* - Q* homeomorphism and investigate their basic properties. Also we define and studied the properties of (τ₁, τ₂)* - irreducible spaces and τ₁τ₂ - irreducible spaces.

The most important property is that the set of all (τ₁, τ₂)* - Q* homeomorphisms is a group under composition of functions.

2. PRELIMINARIES

Throughout this paper X and Y always represent nonempty bitopological spaces (X, τ₁, τ₂) and (Y, σ₁, σ₂). Now we shall require the following known definitions are prerequisites.

Definition 2.1: A subset S of X is called (τ₁, τ₂)*-open if S∈τ₁∪τ₂ and the complement of (τ₁, τ₂)* - open set is (τ₁, τ₂)* - semi open set.

Definition 2.2: A map f: X → Y is called (τ₁, τ₂)* - Q* - continuous if the inverse image of each (σ₁, σ₂)* - Q* closed in Y is τ₁τ₂ - closed in X.

Definition 2.3 [6]: A map f: X → Y is called τ₁τ₂ - Q* - continuous if the inverse image of each σ₁σ₂ - Q* closed in Y is τ₂ - closed in X.

Definition 2.4: A subset S of X is said to be (τ₁, τ₂)*-semi open set if S⊆τ₁τ₂ cl (τ₁τ₂ - int (S)). The complement of (τ₁, τ₂)* - semi open set is (τ₁, τ₂)* - semi closed.

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3. \((\tau_1, \tau_2)^* - Q^*\) HOMEOMORPHISM

Throughout this paper X and Y always represent nonempty bitopological spaces \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\). We denote the family of all \((\tau_1, \tau_2)^* - Q^*\) homeomorphisms from \((X, \tau_1, \tau_2)\) onto itself by \((\tau_1, \tau_2)^* - Q^* H(X)\) and the family of all \((\tau_1, \tau_2)^* - closed set in \((X, \tau_1, \tau_2)\) is denoted by \((\tau_1, \tau_2)^* - C(X)\).

**Definition 3.1[8]:** A bijection \(f: X \to Y\) is called \((\tau_1, \tau_2)^* - Q^*\) homeomorphism, if \(f\) is \((\tau_1, \tau_2)^* - Q^*\) continuous and its inverse also \((\tau_1, \tau_2)^* - Q^*\) continuous.

**Example 3.1:** Let \(X = Y = \{a, b, c\}, \tau_1 = \{\emptyset, X\}, \tau_2 = \{\emptyset, X, \{b\}, \{b, c\}\}\) and \(\sigma_1 = \{\emptyset, Y, \{b\}\}, \sigma_2 = \{\emptyset, Y, \{b\}, \{b, c\}\}\). Then \(f, \phi, [a, c]\) are \((\sigma_1, \sigma_2)^* - Q^*\) closed in \(Y\). Let \(f: X \to Y\) be the identity map. Then \(f(\phi) = \phi, f([a, c]) = [a, c], f([a]) = [a]\). Since \(f, \phi, [a, c], [a]\) are \(\tau_1, \tau_2\) - closed in \(X\). Therefore, \(f\) and \(f^{-1}\) are \((\tau_1, \tau_2)^* - Q^*\) continuous. Hence \(f\) is \((\tau_1, \tau_2)^* - Q^*\) homeomorphism.

**Definition 3.2:** A subset \(S\) of \(X\) is called pairwise \((\tau_1, \tau_2)^* - Q^*\) homeomorphism in \(X\) if \(S\) is both \((\tau_1, \tau_2)^* - Q^*\) homeomorphism and \((\tau_2, \tau_1)^* - Q^*\) homeomorphism.

**Definition 3.3:** A space \((X, \tau_1, \tau_2)\) is called \((\tau_1, \tau_2)^* - Q^*\) space if every \((\tau_1, \tau_2)^* - Q^*\) closed is \((\tau_1, \tau_2)^* - closed.

**Proposition 3.1:** If \(f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) and \(g: (Y, \sigma_1, \sigma_2) \to (Z, \eta_1, \eta_2)\) are \((\tau_1, \tau_2)^* - Q^*\) homeomorphisms, then \(g \circ f: (X, \tau_1, \tau_2) \to (Z, \eta_1, \eta_2)\) is also \((\tau_1, \tau_2)^* - Q^*\) homeomorphism.

**Proof:** Let \(U\) be a \((\eta_1, \eta_2)^* - Q^*\) open set in \((Z, \eta_1, \eta_2)\).

Now \((g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)\), where \(V = g^{-1}(U)\).

By hypothesis, \(V\) is \((\sigma_1, \sigma_2)^* - Q^*\) open in \((Y, \sigma_1, \sigma_2)\) and again by hypothesis, \(f^{-1}(V)\) is \((\tau_1, \tau_2)^* - Q^*\) open in \((X, \tau_1, \tau_2)\).

Therefore, \((g \circ f)^{-1}\) is \((\tau_1, \tau_2)^* - Q^*\) continuous.

Also for a \((\tau_1, \tau_2)^* - Q^*\) open set \(G\) in \((X, \tau_1, \tau_2)\),

We have \((g \circ f)(G) = g(f(G)) = g(W)\), where \(W = f(G)\).

By hypothesis,

\(f(G)\) is \((\sigma_1, \sigma_2)^* - Q^*\) open in \((Y, \sigma_1, \sigma_2)\) and again by hypothesis, \(g(W)\) is \((\eta_1, \eta_2)^* - Q^*\) open in \((Z, \eta_1, \eta_2)\).

Therefore, \((g \circ f)^{-1}\) is \((\tau_1, \tau_2)^* - Q^*\) continuous.

Hence \(g \circ f\) is \((\tau_1, \tau_2)^* - Q^*\) homeomorphism.

**Example 3.2:** Let \(X = Y = Z = \{a, b, c\}, \tau_1 = \{\emptyset, X\}, \tau_2 = \{\emptyset, X, \{b\}, \{b, c\}\}\) and \(\sigma_1 = \{\emptyset, Y, \{b\}\}, \sigma_2 = \{\emptyset, Y, \{b\}, \{b, c\}\}\). Then \(\phi, [a, c]\) are \((\sigma_1, \sigma_2)^* - Q^*\) closed in \(Y\). Let \(f: X \to Y\) be the identity map. Then \(f\) and \(g\) are \((\tau_1, \tau_2)^* - Q^*\) homeomorphic. Here \(g \circ f\) is \((\tau_1, \tau_2)^* - Q^*\) continuous, since \([b, c]\) is \((\eta_1, \eta_2)^* - Q^*\) open in \(Z\) and \((g \circ f)^{-1}([b, c]) = \{b, c\}\) is \((\tau_1, \tau_2)^* - Q^*\) open in \((X, \tau_1, \tau_2)\). Hence \(g \circ f\) is \((\tau_1, \tau_2)^* - Q^*\) homeomorphism.

**Proposition 3.2:** The set \((\tau_1, \tau_2)^* - Q^* H(X)\) is a group.

**Proof:** Define \(\Psi: (\tau_1, \tau_2)^* - Q^* H(X) \times (\tau_1, \tau_2)^* - Q^* H(X) \to (\tau_1, \tau_2)^* - Q^* H(X)\) by \(\Psi(f, g) = (g \circ f)\) for every \(f, g \in (\tau_1, \tau_2)^* - Q^* H(X)\).

Then by proposition 3.1, \((g \circ f) \in (\tau_1, \tau_2)^* - Q^* (X)\).

Hence \((\tau_1, \tau_2)^* - Q^* H(X)\) is closed.

We know that the composition of maps is associative.

The identity map \(i: (\tau_1, \tau_2)^* - Q^* H(X) \to (\tau_1, \tau_2)^* - Q^* H(X)\) is a \((\tau_1, \tau_2)^* - Q^* H(X)\).

Also \(i \circ i = f \circ f = f\) for every \(f \in (\tau_1, \tau_2)^* - Q^* H(X)\).

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For any \( f \in (\tau_1, \tau_2)^* - Q^* H(X) \),
\[ f \circ f^{-1} = f^{-1} \circ f = i. \]

Hence inverse exists for each element of \((\tau_1, \tau_2)^* - Q^* H(X)\).

Thus, \((\tau_1, \tau_2)^* - Q^* H(X)\) is a group under composition of maps.

**Theorem 3.1:** Every \((\tau_1, \tau_2)^* - Q^* \) homeomorphism is a \((\tau_1, \tau_2)^* - \) homeomorphism.

**Proof:** Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a \((\tau_1, \tau_2)^* - Q^* \) homeomorphism.

Then \( f \) is bijective and both \( f \) and \( f^{-1} \) are \((\tau_1, \tau_2)^* - Q^* \) continuous.

Since every \((\tau_1, \tau_2)^* - Q^* \) continuous function is \((\tau_1, \tau_2)^* \) continuous we have \( f \) and \( f^{-1} \) are \((\tau_1, \tau_2)^* - \) continuous.

This shows that \( f \) is a \((\tau_1, \tau_2)^* - \) homeomorphism.

**Remark 3.1:** The converse of the above theorem need not be true, as shown in the following example.

**Example 3.3:** In example 3.1, \( f \) is \((\tau_1, \tau_2)^* - Q^* \) homeomorphism but not \((\tau_1, \tau_2)^* - \) homeomorphism.

**Proposition 3.3:** If \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) and \( g: (Y, \sigma_1, \sigma_2) \rightarrow (Z, \eta_1, \eta_2) \) are \((\tau_1, \tau_2)^* - Q^* \) homeomorphisms, then \( g \circ f: (X, \tau_1, \tau_2) \rightarrow (Z, \eta_1, \eta_2) \) is also a \((\tau_1, \tau_2)^* - Q^* \) homeomorphism.

**Proof:** Let \( U \) be a \( \eta_1, \eta_2 - Q^* \) open set in \((Z, \eta_1, \eta_2)\).

Now \( (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V) \), where \( V = g^{-1}(U) \).

By hypothesis, \( V \) is \( \sigma_1, \sigma_2 - Q^* \) open in \((Y, \sigma_1, \sigma_2)\) and again by hypothesis, \( f^{-1}(V) \) is \( \tau_1, \tau_2 - Q^* \) open in \((X, \tau_1, \tau_2)\).

Therefore, \( (g \circ f) \) is \( \tau_1, \tau_2 - Q^* \) continuous.

Also for a \( \tau_1, \tau_2 - Q^* \) open set \( G \) in \((X, \tau_1, \tau_2)\),
we have \( (g \circ f)(G) = g(f(G)) = g(W) \), where \( W = f(G) \).

By hypothesis,
\( f(G) \) is \( \sigma_1, \sigma_2 - Q^* \) open in \((Y, \sigma_1, \sigma_2)\) and again by hypothesis, \( g(W) \) is \( \eta_1, \eta_2 - Q^* \) open in \((Z, \eta_1, \eta_2)\).

Therefore, \( (g \circ f)^{-1} \) is \( \tau_1, \tau_2 - Q^* \) continuous.

Hence \( g \circ f \) is \( \tau_1, \tau_2 - Q^* \) homeomorphism.

**Example 3.4:** Let \( X = Y = Z = \{a, b, c\}, \tau_1 = \{\phi, X\}, \tau_2 = \{\phi, X, \{a\}, \{a, c\}\} \) and \( \sigma_1 = \{\phi, Y, \{a\}\}, \sigma_2 = \{\phi, Y, \{a, a, c\}\}, \eta_1 = \{\phi, Z, \{a\}, \{a, c\}\}, \eta_2 = \{\phi, Z, \{a, c\}\} \).

Then \( \phi \) is \( \tau_1, \tau_2 - Q^* \) closed in \( Y \). Let \( f: X \rightarrow Y \) be the identity map. Then \( f \) and \( g \) are \( \tau_1, \tau_2 - Q^* \) homeomorphism . Here \( g \circ f \) is \( \tau_1, \tau_2 - Q^* \) continuous, since \( \{a, c\} \) is \( \eta_1, \eta_2 - Q^* \) open in \( Z \) and \( (g \circ f)^{-1}(\{a, c\}) = \{a, c\} \) is \( \tau_1, \tau_2 - Q^* \) open in \((X, \tau_1, \tau_2)\). Hence \( g \circ f \) is \( \tau_1, \tau_2 - Q^* \) homeomorphism.

**Proposition 3.4:** The set \( \tau_1, \tau_2 - Q^* H(X, \tau_1, \tau_2) \) is a group.

**Proof:** Define \( \Psi: \tau_1, \tau_2 - Q^* H(X, \tau_1, \tau_2) \times \tau_1, \tau_2 - Q^* H(X, \tau_1, \tau_2) \rightarrow \tau_1, \tau_2 - Q^* H(X, \tau_1, \tau_2) \) by \( \Psi(f, g) = (g \circ f) \) for every \( f, g \in \tau_1, \tau_2 - Q^* H(X, \tau_1, \tau_2) \).

Then by proposition 3.3, \( (g \circ f) \in \tau_1, \tau_2 - Q^* H(X, \tau_1, \tau_2) \).

Hence \( \tau_1, \tau_2 - Q^* H(X, \tau_1, \tau_2) \) is closed.

We know that the composition of maps is associative.
The identity map
\[ i: (X, \tau_1, \tau_2) \rightarrow (X, \tau_1, \tau_2) \] is a \( \tau_1 \tau_2 \)-\( Q^* \) homeomorphism and \( i \in \tau_1 \tau_2 \)-\( Q^* \)H(X, \tau_1, \tau_2) .

Also \( i \circ f = f \circ i = f \) for every \( f \in \tau_1 \tau_2 \)-\( Q^* \)H(X, \tau_1, \tau_2).

For any \( f \in \tau_1 \tau_2 \)-\( Q^* \)H(X, \tau_1, \tau_2),
\[ f \circ i = f^{-1} \circ f = i. \]

Hence inverse exists for each element of \( \tau_1 \tau_2 \)-\( Q^* \)H(X, \tau_1, \tau_2).

Thus, \( \tau_1 \tau_2 \)-\( Q^* \)H(X, \tau_1, \tau_2) is a group under composition of maps.

**Theorem 3.2** - Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a map. Then the following are true.

i) Every \( \tau_1 \tau_2 \)-\( Q^* \) homeomorphism is \( \tau_1 \tau_2 \)-homeomorphism.

ii) Every \( (\tau_1, \tau_2)^* \)-\( Q^* \) homeomorphism is \( \tau_1 \tau_2 \)-\( Q^* \) homeomorphism.

**Proof:** The proof is obvious.

**Definition 3.4:** For a subset \( A \) of a space \( (X, \tau_1, \tau_2) \) we define the \( (\tau_1, \tau_2)^* \)-\( Q^* \) kernel of \( A \) (briefly, \( (\tau_1, \tau_2)^* \)-\( Q^* \) ker(A)) as follows: \( (\tau_1, \tau_2)^* \)-\( Q^* \) ker(A) = \( \cap \{ F: F \in (\tau_1, \tau_2)^* \}-\( Q^* \) O(X, \tau_1, \tau_2); A \subset F \}. \) \( A \) is said to be a \( (\tau_1, \tau_2)^* \)-\( Q^* \) set in \( (X, \tau_1, \tau_2) \) if \( A = (\tau_1, \tau_2)^* \)-\( Q^* \) ker(A), or equivalently, if \( A \) is the intersection of \( (\tau_1, \tau_2)^* \)-\( Q^* \) open sets. \( A \) is said to be \( (\tau_1, \tau_2)^* \)-\( Q^* \) close in \( (X, \tau_1, \tau_2) \) if it is the intersection of a \( (\tau_1, \tau_2)^* \)-\( Q^* \) \( \lambda \) set in \( (X, \tau_1, \tau_2) \) and a quasi closed set in \( (X, \tau_1, \tau_2) \). Clearly, \( (\tau_1, \tau_2)^* \)-\( Q^* \) - \( \lambda \) set sand \( (\tau_1, \tau_2)^* \)-\( Q^* \) closed sets are \( (\tau_1, \tau_2)^* \)-\( Q^* - \lambda \) closed; complements of \( (\tau_1, \tau_2)^* \)-\( Q^* \) - \( \lambda \) closed sets in \( (X, \tau_1, \tau_2) \) are said to be \( (\tau_1, \tau_2)^* \)-\( Q^* \) open in \( (X, \tau_1, \tau_2) \).

**Proposition 3.5:** For a subset \( A \) of a space \( (X, \tau_1, \tau_2) \), the following are equivalent:

(i) \( A \) is \( (\tau_1, \tau_2)^* \)-\( Q^* \) - \( \lambda \) closed in \( (X, \tau_1, \tau_2) \).

(ii) \( A = L \cap (\tau_1, \tau_2)^* \)-\( Q^* \) cl(A), where \( L \) is a \( (\tau_1, \tau_2)^* \)-\( Q^* - \lambda \) set in \( (X, \tau_1, \tau_2) \).

(iii) \( A = (\tau_1, \tau_2)^* \)-\( Q^* \) ker(A) \( \cap (\tau_1, \tau_2)^* \)-\( Q^* \) cl(A).

**Definition 3.5:** A bijection \( f: X \rightarrow Y \) is called \( (\tau_1, \tau_2)^* \)-\( Q^* \) homeomorphism, if \( f \) is \( (\tau_1, \tau_2)^* \)-\( Q^* \) irresolute and its inverse also \( (\tau_1, \tau_2)^* \)-\( Q^* \) irresolute.

**Remark 3.2:** We say that spaces \( (X, \tau_1, \tau_2) \) and \( (Y, \sigma_1, \sigma_2) \) are \( (\tau_1, \tau_2)^* \)-\( Q^* \) homeomorphic if there exists a \( (\tau_1, \tau_2)^* \)-\( Q^* \) homeomorphism from \( (X, \tau_1, \tau_2) \) onto \( (Y, \sigma_1, \sigma_2) \).

**Theorem 3.3:** Every \( (\tau_1, \tau_2)^* \)-\( Q^* \) homeomorphism is a \( (\tau_1, \tau_2)^* \)-\( Q^* \) homeomorphism.

**Proof:** Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a \( (\tau_1, \tau_2)^* \)-\( Q^* \) homeomorphism.

Then \( f \) is bijective, \( (\tau_1, \tau_2)^* \)-\( Q^* \) irresolute and \( f^{-1} \) is \( (\tau_1, \tau_2)^* \)-\( Q^* \) irresolute.

Since every \( (\tau_1, \tau_2)^* \)-\( Q^* \) irresolute is \( (\tau_1, \tau_2)^* \)-\( Q^* \) continuous, \( f \) and \( f^{-1} \) are \( (\tau_1, \tau_2)^* \)-\( Q^* \) continuous and so \( f \) is a \( (\tau_1, \tau_2)^* \)-\( Q^* \) homeomorphism.

**Remark 3.3:** The following example shows that the converse of the above theorem need not be true.

**Example 3.5:** Let \( X = Y = \{ a, b, c \} \), \( \tau_1 = \{ \phi, X, \{ a \}, \{ c \}, \{ a, c \} \} \) and \( \tau_2 = \{ \phi, X, \{ a \} \} \). Clearly \( \{ b \} \) is \( (\tau_1, \tau_2)^* \)-\( Q^* \) closed in \( X \). Let \( \sigma_1 = \{ \phi, Y, \{ a \} \} \) and \( \sigma_2 = \{ \phi, Y \} \).

Then \( \sigma_1 \sigma_2 \) - open sets on \( Y \) are \( \phi, Y, \{ a \} \) and \( \sigma_1 \sigma_2 \) - closed sets on \( X \) are \( \{ b, c \} \). Since \( \{ b, c \} \) is \( (\sigma_1, \sigma_2)^* \)-\( Q^* \) closed in \( Y \) but \( f^{-1}(\{ b, c \}) = \{ b, c \} \) is not \( (\tau_1, \tau_2)^* \)-\( Q^* \) open in \( X \) and so \( f \) is not \( (\tau_1, \tau_2)^* \)-\( Q^* \) irresolute.

**Remark 3.4:** The above example shows that the concepts of \( (\tau_1, \tau_2)^* \)-homeomorphisms and \( (\tau_1, \tau_2)^* \)-\( Q^* \)homeomorphisms are independent.

**Definition 3.6:** A map \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is said to be \( (\tau_1, \tau_2)^* \)-\( Q^* \) closed if for every \( (\tau_1, \tau_2)^* \)-\( Q^* \) closed \( F \) of \( X \), \( f(F) \) is \( (\tau_1, \tau_2)^* \)-closed in \( Y \).
**Proposition 3.6** - For any bijection \( f: X \rightarrow Y \), the following statements are equivalent.
(a) \( f^{-1}: Y \rightarrow X \) is \( (\tau_1, \tau_2)^* - Q^* \) continuous.
(b) \( f \) is a \( (\tau_1, \tau_2)^* - Q^* \) open map.
(c) \( f \) is a \( (\tau_1, \tau_2)^* - Q^* \) closed map.

**Proof:**

**Step 1:** (a) \( \Rightarrow \) (b)
Let \( V \) be a \( (\tau_1, \tau_2)^* \) - open set in \( X \).

Then \( X - V \) is \( (\tau_1, \tau_2)^* \) - closed in \( X \).

Since \( f^{-1} \) is \( (\tau_1, \tau_2)^* - Q^* \) continuous,

\( (f^{-1})^{-1}(X - V) = f(X - V) = Y - f(V) \) is \( (\tau_1, \tau_2)^* - Q^* \) closed in \( Y \).

Then \( f(V) \) is \( (\tau_1, \tau_2)^* - Q^* \) open in \( Y \).

Hence \( f \) is a \( (\tau_1, \tau_2)^* - Q^* \) open map.

**Step 2:** (b) \( \Rightarrow \) (c).
Let \( f \) be a \( (\tau_1, \tau_2)^* - Q^* \) open map.

Let \( U \) be \( (\tau_1, \tau_2)^* \) - closed set in \( X \).

Then \( X - U \) is \( (\tau_1, \tau_2)^* \) - open in \( X \).

Since \( f \) is \( (\tau_1, \tau_2)^* - Q^* \) open,

\( f(X - U) = Y - f(U) \) is \( (\tau_1, \tau_2)^* - Q^* \) open in \( Y \).

Then \( f(U) \) is \( (\tau_1, \tau_2)^* - Q^* \) closed in \( Y \).

Hence \( f \) is a \( (\tau_1, \tau_2)^* - Q^* \) closed.

**Step 3:** (c) \( \Rightarrow \) (a).
Let \( V \) be \( (\tau_1, \tau_2)^* - Q^* \) closed set in \( X \).

Since \( f: X \rightarrow Y \) is \( (\tau_1, \tau_2)^* - Q^* \) closed,

\( f(V) \) is \( (\tau_1, \tau_2)^* - Q^* \) closed in \( Y \).

That is \( (f^{-1})^{-1}(V) \) is \( (\tau_1, \tau_2)^* - Q^* \) closed in \( Y \).

Hence \( f^{-1} \) is \( (\tau_1, \tau_2)^* - Q^* \) continuous.

**Proposition 3.7:** Let \( f: X \rightarrow Y \) be a bijective and \( (\tau_1, \tau_2)^* - Q^* \) continuous map. Then the following statements are equivalent.
(a) \( f \) is a \( (\tau_1, \tau_2)^* - Q^* \) open map.
(b) \( f \) is a \( (\tau_1, \tau_2)^* - Q^* \) homeomorphism.
(c) \( f \) is a \( (\tau_1, \tau_2)^* - Q^* \) closed map.

**Proof:**

**Step 1:** (a) \( \Rightarrow \) (b).
Given \( f \) is bijective, \( (\tau_1, \tau_2)^* - Q^* \) continuous map and \( (\tau_1, \tau_2)^* - Q^* \) open map. Hence \( f \) is \( (\tau_1, \tau_2)^* - Q^* \) homeomorphism.

**Step 2:** (b) \( \Rightarrow \) (c).
Let \( f \) be a \( (\tau_1, \tau_2)^* - Q^* \) homeomorphism.

Hence \( f \) is \( (\tau_1, \tau_2)^* - Q^* \) open.
By Proposition 3.6, \( f \) is \((\tau_1, \tau_2)* - Q^*\) closed.

**Step 3:** (c) \(\Rightarrow\) (a)

Follows from Proposition 3.6.

**Definition 3.7:** Let \( S \) be a subset of \( X \). Let \( x \in X \). Then \( x \) is said to be a \((\tau_1, \tau_2)* - Q^*\) limit point of \( S \) if and only if every \((\tau_1, \tau_2)* - Q^*\) open set containing \( x \) contains at least one point other than \( x \).

**Definition 3.8:** Let \( S \) be a subset of \( X \). Then the set of all \((\tau_1, \tau_2)* - Q^*\) limit points of \( S \) is said to be \((\tau_1, \tau_2)* - Q^*\) derived set of \( S \) and it is denoted by \((\tau_1, \tau_2)* - D Q^* (S)\).

**Theorem 3.4:** Let \( A \) be a subset of \( X \). Let \((\tau_1, \tau_2)* - D Q^* (A)\) be the set of all \((\tau_1, \tau_2)* - Q^*\) limit points of \( A \). Then \((\tau_1, \tau_2)* - Q^* cl(A) = A \cup (\tau_1, \tau_2)* - D Q^* (A)\).

**Proof:** Let \( x \in A \cup (\tau_1, \tau_2)* - D Q^* (A)\).

This implies either \( x \in A \) or \( x \in (\tau_1, \tau_2)* - D Q^* (A)\).

If \( x \in A \), then \( x \in (\tau_1, \tau_2)* - Q^* cl(A)\).

If \( x \in (\tau_1, \tau_2)* - D Q^* (A)\), then every \((\tau_1, \tau_2)* - Q^*\) open set contains \( x \) will intersect with \( A \).

Therefore, \( x \in (\tau_1, \tau_2)* - Q^* cl(A)\).

This implies \( A \cup (\tau_1, \tau_2)* - D Q^* (A) \subseteq (\tau_1, \tau_2)* - Q^* cl(A)\).

If \( x \in (\tau_1, \tau_2)* - Q^* cl(A)\), then to prove \( x \in A \cup (\tau_1, \tau_2)* - D Q^* (A)\).

If \( x \in A \), then \( x \in A \cup (\tau_1, \tau_2)* - D Q^* (A)\).

If \( x \notin A \), since \( x \in (\tau_1, \tau_2)* - Q^* cl(A)\) implies every \((\tau_1, \tau_2)* - Q^*\) open set of \( x \) intersects with \( A \).

Hence \( x \in (\tau_1, \tau_2)* - D Q^* (A)\).

Therefore, \((\tau_1, \tau_2)* - Q^* cl(A) = A \cup (\tau_1, \tau_2)* - D Q^* (A)\).

**Definition 3.9:** Let \( S \) be a subset of \( X \). Any point of \((\tau_1, \tau_2)* - Q^* cl (S)\) is referred to as a \((\tau_1, \tau_2)* - Q^*\) contact (or adherent) point of \( S \).

**Definition 3.10 [6]:** A bijection \( f: X \rightarrow Y \) is called \((\tau_1, \tau_2)* - Q^*\) homeomorphism, if \( f \) is \((\tau_1, \tau_2)* - Q^*\) continuous and its inverse also \((\tau_1, \tau_2)* - Q^*\) continuous.

**Example 3.6:** In example 3.1, \( \phi, \{a\}, \{a, c\} \) are \( \sigma_1 \sigma_2 - Q^*\) closed in \( Y \). Let \( f: X \rightarrow Y \) be the identity map. Then \( f(\phi) = \phi, f(\{a\}, \{a, c\}, f(\{a\}) = \{a\} \). Since \( \phi, \{a, c\}, \{a\} \) are \( \tau_2\) - closed in \( X \). Therefore, \( f \) and \( f^{-1} \) are \( \tau_1 \tau_2 - Q^*\) continuous.

Hence \( f \) is a \( \tau_1 \tau_2 - Q^*\) homeomorphism.

**Definition 3.11:** A bijection \( f: X \rightarrow Y \) is called \( \tau_1 \tau_2 - Q^*\) homeomorphism, if \( f \) is \( \tau_1 \tau_2 - Q^*\) irresolute and its inverse also \( \tau_1 \tau_2 - Q^*\) irresolute.

**Theorem 3.3:** Every \( \tau_1 \tau_2 - Q^*\) homeomorphism is a \( \tau_1 \tau_2 - Q^*\) homeomorphism.

**Proof:** Let \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a \( \tau_1 \tau_2 - Q^*\) homeomorphism.

Then \( f \) is bijective, \( \tau_1 \tau_2 - Q^*\) irresolute and \( f^{-1} \) is \( \tau_1 \tau_2 - Q^*\) irresolute.

Since every \( \tau_1 \tau_2 - Q^*\) irresolute is \( \tau_1 \tau_2 - Q^*\) continuous, \( f \) and \( f^{-1} \) are \( \tau_1 \tau_2 - Q^*\) continuous and so \( f \) is a \( \tau_1 \tau_2 - Q^*\) homeomorphism.

**Remark 3.3:** The following example shows that the converse of the above theorem need not be true.
Example 3.7: Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}, \{a, c\}\}$ and $\tau_2 = \{\emptyset, X, \{a\}\}$. Clearly $\{b, c\}$ is $\tau_1 \tau_2$-Q* closed in $X$. Let $\sigma_1 = \{\emptyset, Y, \{a\}\}$ and $\sigma_2 = \{\emptyset, Y\}$. Then $\sigma_1 \sigma_2$-open sets on $Y$ are $\emptyset, Y, \{a\}$ and $\sigma_1 \sigma_2$-closed sets on $X$ are $\emptyset, Y, \{b, c\}$. Since $\{b, c\}$ is $\sigma_1 \sigma_2$-Q* closed in $Y$ but $f^{-1}(\{b, c\}) = \{b, c\}$ is not $\tau_1 \tau_2$-Q* open in $X$ and so $f$ is not $\tau_1 \tau_2$-Q* irresolute.

Remark 3.4: The above example shows that the concepts of $\tau_1 \tau_2$-homeomorphisms and $\tau_1 \tau_2$-Q* homeomorphism are independent.

Definition 3.12: A bitopological space $(X, \tau_1, \tau_2)$ is called $(\tau_1, \tau_2)$*-irreducible if $X$ is not empty and whenever $X = A_1 \cup A_2$ with $(\tau_1, \tau_2)$*-closed subsets $A_i \in X$ ($i = 1, 2$) then we have $X = A_1$ or $A_2$.

Example: Let $X = \{1, 2, 3\}$ and $\tau_1 = \{\emptyset, X, \{1\}, \{1, 2\}\}$ and $\tau_2 = \{\emptyset, X, \{1\}\}$. Then $(\tau_1, \tau_2)$*-closed sets are $\emptyset, X, \{2, 3\}, \{3\}$. Then $X$ is $(\tau_1, \tau_2)$*-irreducible.

Theorem: A bitopological space $X$ is $(\tau_1, \tau_2)$*-irreducible if and only if every nonempty open set is $(\tau_1, \tau_2)$*-Q* open.

Proof: Let $X$ be a $(\tau_1, \tau_2)$*-irreducible.

Let $U$ be any nonempty open set.

If $U = X$ then nothing to prove.

Let $U \neq X$.

Then $(\tau_1, \tau_2)$*-cl $(U) \neq X$.

Then there exists an $(\tau_1, \tau_2)$*-open set $V$ such that $U \cap V = \emptyset$.

This implies $U^c \cap V^c = X$, where $U^c$ and $V^c$ are proper $(\tau_1, \tau_2)$*-closed sets which is a contradiction to the fact that $X$ is $(\tau_1, \tau_2)$*-irreducible.

Conversely assume that every $(\tau_1, \tau_2)$*-open set is $(\tau_1, \tau_2)$*-Q* open.

We claim that $X$ is $(\tau_1, \tau_2)$*-irreducible.

Then $X = A \cup B$, where $A$ and $B$ are proper nonempty $(\tau_1, \tau_2)$*-closed sets.

$A^c \cap B^c = \emptyset$.

Then $A^c$ is not dense.

Then $A^c$ is an $(\tau_1, \tau_2)$*-open set but not $(\tau_1, \tau_2)$*- Q* open.

Hence $X$ is irreducible.

Definition 3.13: A bitopological space $(X, \tau_i, \tau_j)$ is called $\tau_i \tau_j$-irreducible if $X$ is not empty and whenever $X = A_1 \cup A_2$ with $\tau_i$-closed subset $A_1 \in X$ and $\tau_j$-closed subset $A_2 \in X$ ($i, j = 1, 2$) then we have $X = A_1$ or $A_2$.

Example: Let $X = \{1, 2, 3\}$ and $\tau_1 = \{\emptyset, X, \{1\}, \{1, 3\}\}$ and $\tau_2 = \{\emptyset, X, \{2\}, \{2, 3\}\}$. Then $\tau_1$-closed sets are $\emptyset, X, \{2, 3\}, \{2\}$ and $\tau_2$-closed sets are $\emptyset, X, \{1, 3\}, \{1\}$. Then $X$ is $\tau_1 \tau_2$-irreducible.

Theorem: A bitopological space $X$ is $\tau_1 \tau_2$-irreducible if and only if every nonempty open set is $\tau_1 \tau_2$- Q* open.

Proof: Let $X$ be a $\tau_1 \tau_2$-irreducible.

Let $U$ be any nonempty $\tau_j$-open set.

If $U = X$ then nothing to prove.

Let $U \neq X$. 

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Then $\tau_i - \text{cl} (U) \neq X$. Then there exists an $\tau_i$- open set $V$ such that $U \cap V = \phi$.

This implies $U^c \cap V^c = X$, where $U^c$ and $V^c$ are proper $\tau_i$ - closed set and $\tau_j$-closed set which is a contradiction to the fact that $X$ is $\tau_i \tau_j$ - irreducible.

Conversely assume that every $\tau_i \tau_j$ - open set is $\tau_i \tau_j$ - $Q^*$ open.

We claim that $X$ is $\tau_i \tau_j$ - irreducible.

Then $X = A \cup B$, where $A$ and $B$ are proper nonempty $\tau_i$ - closed set and $\tau_j$ - closed set.

$A^c \cap B^c = \phi$.

Then $A^c$ is not dense.

Then $A^c$ is an $\tau_i \tau_j$ - open set but not $\tau_i \tau_j$ - $Q^*$ open.

Hence $X$ is irreducible.

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