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ON NORMAL ALMOST DISTRIBUTIVE LATTICES

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ABSTRACT

We examine the normality of subADLs of a normal ADL_0. We obtain necessary and sufficient conditions for an ADL_0 to become normal (relatively normal) in terms of filter congruences and prime filter congruences.

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INTRODUCTION

The concept of almost distributive lattice with zero (ADL_0) was introduced by Swamy and Rao [5] in 1980 as a common abstract of ring theoretic and lattice theoretic generalization of Boolean algebras. It is an algebraic structure of type (2, 2, 0) which satisfies all the conditions of a distributive lattice except the commutativity of $\vee$, $\wedge$ and the right distributivity of $\vee$ over $\wedge$. Rao and Ravi Kumar [3] introduced the concept of the normality of an ADL_0 in 2008. They obtained several equivalent conditions for an ADL_0 to become a normal almost distributive lattice in terms of prime ideals, minimal prime ideal and annihilator ideals.

In this paper, we observe that a subADL_0 of a normal ADL_0 need not be normal. We obtain a necessary and sufficient condition for a subADL_0 of a normal ADL_0 to become a normal subADL_0. We study the normality and relative normality of an ADL_0 in terms of filter congruences and prime filter congruences

1. PRELIMINARIES

First we recall the definitions and certain necessary properties of almost distributive lattices with zero from [5].

Definition 1.1: [5] An Algebra $(L, \vee, \wedge, 0)$ of type (2, 2, 0) is called an almost distributive lattice with 0 (ADL_0) if, it satisfies the following conditions.

(i) $0 \wedge a = 0$
(ii) $a \vee 0 = a$
(iii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(iv) $(a \vee b) \wedge c = (a \wedge c) \vee (a \wedge b)$
(v) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(vi) $(a \vee b) \wedge b = b$

for all a, b, c $\in$ L.

Example 1.2: [5] Let X be a non empty set. Fix $x_0 \in X$. For any x, y $\in$ X, define

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \quad \text{and} \quad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x & \text{if } x = x_0 \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an almost distributive lattice with x_0 as its " 0 "

Form here onwards L means almost distributive lattice with '0' as its zero element. For any a, b $\in$ L, we say that a is less than or equal to b (that is, $a \leq b$) if $a \wedge b = a$ or equivalently $a \vee b = b$. It can be easily verified that ' $\leq$ ' is a partial ordering on L. An element m of L is said to be maximal if $m \wedge x = x$ for all $x \in L$.

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Definition 1.3: [5] A non empty sub set I of L is said to be an ideal (filter) of L, if it satisfies the following conditions;
(i) For all a, b $\in$ L, a $\vee$ b $\in$ L (a $\wedge$ b $\in$ L)
(ii) For all a $\in$ L, x $\in$ I, a $\wedge$ x $\in$ I (x $\vee$ a $\in$ L)

A proper ideal (filter) P of L is said to be a prime ideal (filter) if, for any a, b $\in$ L, a $\wedge$ b $\in$ P (a $\vee$ b $\in$ P) implies a $\in$ P or b $\in$ P. It can be routinely verified that a proper sub set P of L is prime ideal of L if and only if L-P is a prime filter of L.

Definition 1.4: [5] A prime ideal P of L is said to be a minimal prime ideal of L, if there is no prime ideal which is properly contained in P. Similarly, a proper filter P of L is said to be maximal filter of L if there is no proper filter containing P. It can be easily verified that a proper ideal P of L is a minimal prime ideal of L if and only if L-P is a maximal filter of L. Since every proper filter contained in a maximal filter, every non-zero element is contained in a maximal filter. Therefore for any non-zero element x of L, there is a minimal prime ideal P of L such that x $\notin$ P. Hence we have the following.

Theorem 1.5: [6] The intersection of all minimal prime ideals of L is equal to {0}.

For any x $\in$ L, the set $(x)^* = \{y \in L \mid x \wedge y = 0\}$ is an ideal of L.

Theorem 1.6:[4] A prime ideal P of L is minimal if and only if, for each x $\in$ P, there exists y $\notin$ P such that x $\wedge$ y = 0. (That is, (L-P) $\bigcap (x)^*$ is non-empty.)

Definition 1.7: [5] L is said to be a relatively complemented if, given a, b $\in$ L, there exists x $\in$ L such that a $\wedge$ x = 0 and a $\vee$ x = a $\vee$ b.

Theorem 1.8: [5] L is relatively complemented if and only if every prime ideal is minimal.

2. ON NORMAL ALMOST DISTRIBUTIVE LATTICES

An almost distributive lattice with zero is called normal [3] if, every prime ideal contains a unique minimal prime ideal or equivalently, for x, y in L, x $\wedge$ y = 0 implies $(x)^* \vee (y)^* = L$. A non empty subset S of L is said to be a subADL$_0$, if it contains “0” and closed under operations $\vee$ and $\wedge$. An almost distributive lattice with zero L is said to be a dense if {0} is a prime ideal of L.

We observe that a subADL$_0$ of a normal ADL$_0$ need not be normal. For, consider the following example.

Example 2.1: Let X = {a, b, c}. Let L = {ϕ, {a}, {b}, {a, b}, X}. Then L is a subADL$_0$ of P(X) with respect to the set inclusion and L is not normal.

In this context, we obtain the following.

Theorem 2.2: The following are equivalent for any ADL$_0$ L.
(i) Every subADL$_0$ of L is normal
(ii) For x, y $\in$ L-{0}, x $\wedge$ y = 0, implies x $\vee$ y is maximal
(iii) L is dense ADL$_0$ or L relatively complemented and every chain in L has at most three elements.

Proof: (i) $\Rightarrow$ (ii): Suppose that every subADL$_0$ of L is normal. Let x, y $\in$ L-{0} such that x $\wedge$ y = 0. Suppose there is z $\in$ L such that x $\vee$ y < z. Then L_1 = {0, x, y, x $\vee$ y, z} is a subADL$_0$ of L which is not normal. This is a contradiction to our assumption. Therefore x $\vee$ y is maximal.

(ii) $\Rightarrow$ (iii): Assume **(ii)**. Suppose L is not dense ADL$_0$. Then {0} is not a prime ideal of L. Let P be a prime ideal of L. Suppose P is not a minimal prime ideal of L. Then there is a minimal prime ideal M ($\neq$ {0}) of L such that M $\subset$ P. Choose x $\in$ M such that x $\neq$ 0. Now, for any y $\in$ L, 0 $\neq$ y $\in$ $(x)^* \bigcap P \Rightarrow x \wedge y = 0$ and y $\in$ P $\Rightarrow$ x $\vee$ y is maximal (by our assumption) and x $\vee$ y $\in$ P (since y $\in$ P and x $\in$ M $\subset$ P). This is a contradiction to the minimality of M (see Theorem 1.6). Therefore every prime ideal is minimal and hence L is relatively complemented (see Theorem 1.8). Let x, y, z $\in$ L-{0} such that 0 < x < y < z. Since L is relatively complemented, there exists t $\in$ L such that x $\wedge$ t = 0 and x $\vee$ t = x $\vee$ y = y. By our assumption, we get that y = x $\vee$ t is maximal. This is a contradiction. Therefore every chain has at most three elements.

(iii) $\Rightarrow$ (i): Assume **(iii).** Let L_1 be a subADL$_0$ of L. Let x, y $\in$ L_1 such that x $\wedge$ y = 0. Suppose x $\neq$ 0 and y $\neq$ 0.

Then 0 < x < x $\vee$ y. Otherwise x = x $\vee$ y implies y = (x $\vee$ y) $\wedge$ y = x $\wedge$ y = 0, which is a contradiction. By our assumption, x $\vee$ y is maximal in L_1 and x $\vee$ y $\in (y)^*_{L1} \vee (x)^*_{L1} = L_1$. Thus L_1 is normal.

Given a filter F of L, define φ_F: = {(a, b) $\in$ L $\times$ L | x $\wedge$ a = x $\wedge$ b for some x $\in$ F}. Then φ_F is a congruence on L.

The following is a routine verification.

Theorem 2.3: For any filter F of L and $x \in$ L, we have

(i) $\frac{x}{\varphi_F}$ is a maximal in $\frac{L}{\varphi_F}$ if and only if $x \in$ F

(ii) $\frac{x}{\varphi_F} = \frac{0}{\varphi_F}$ if and only if $(x)^* \bigcap F \neq \phi$

Proof: (i) Suppose $\frac{x}{\varphi_F}$ is a maximal element in $\frac{L}{\varphi_F}$. Since F $\neq \phi$, we can choose y $\in$F. Then $\frac{x}{\varphi_F} \wedge \frac{y}{\varphi_F} = \frac{y}{\varphi_F}$. That is (x$\wedge$y, y) $\in\varphi_F$. Therefore there exists a $\in$F such that a $\wedge$ x $\wedge$ y = a $\wedge$ y $\in$ F and hence x $\in$ F. On the other hand, suppose x $\in$ F. Let y $\in$ L. The x $\wedge$ x $\wedge$ y = x $\wedge$ y.

Therefore (x $\wedge$ y, y) $\in \varphi_F$ and hence $\frac{x}{\varphi_F} \wedge \frac{y}{\varphi_F} = \frac{y}{\varphi_F}$. Thus $\frac{x}{\varphi_F}$ is a maximal element in $\frac{L}{\varphi_F}$.

(iii) Suppose $\frac{x}{\varphi_F} = \frac{0}{\varphi_F}$. Then there exists $y \in F$ such that y $\wedge$ x = y $\wedge$ 0 = 0. Therefore $y \in (x)^* \bigcap F$ and hence $(x)^* \bigcap F \neq \phi$. On the other hand, suppose $(x)^* \bigcap F \neq \phi$. Choose $y \in F$ such that $x \wedge y =$ 0. Then $y \wedge x = y \wedge$ 0. Therefore (x, 0) $\in\varphi_F$ and hence $\frac{x}{\varphi_F} = \frac{0}{\varphi_F}$.

In the following, for any filter F of L, we obtain a one-to-one correspondence between prime ideals of L disjoint with F and prime ideal of $\frac{L}{\varphi_F}$.

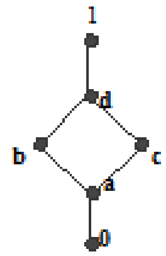
Theorem 2.4: Let F be a filter of L. For any prime ideals P of L with P $\bigcap$ F = ϕ, let $\overline{P} = \left\{ \frac{x}{\varphi_F} \in \frac{L}{\varphi_F} \,|\, x \in P \right\}$. Then $\overline{P}$ is a prime ideal of $\frac{L}{\varphi_F}$. Also, $P \mapsto \overline{P}$ is an order isomorphism (with respect to the inclusion ordering) of the set of prime ideals of L disjoint with F onto the set of prime ideals of $\frac{L}{\varphi_F}$. This map induces a one to one correspondence between minimal prime ideals of L disjoint with F and minimal prime ideals of $\frac{L}{\varphi_F}$.

Proof: Let P be a prime ideal of L such that $P \bigcap F = \phi$. Then, it is easily to verify that $\overline{P} = \left\{ \frac{x}{\varphi_F} \,|\, x \in P \right\}$ is an ideal of $\frac{L}{\varphi_F}$. Also, we observe that, for any $a \in L$, $\frac{a}{\varphi_F} \in \overline{P} \Leftrightarrow a \in P$. Now, for any $x, y \in L$,

$\frac{x}{\varphi_F} \wedge \frac{y}{\varphi_F} \in \overline{P} \Rightarrow \frac{(x \wedge y)}{\varphi_F} \in \overline{P} \Rightarrow x \wedge y \in P \Rightarrow x \in P$ or $y \in P$ (since P is prime) $\Rightarrow \frac{x}{\varphi_F} \in \overline{P}$ or $\frac{y}{\varphi_F} \in \overline{P}$.

Therefore $\overline{P}$ is a prime ideal of $\frac{L}{\varphi_F}$. Now, for any prime ideals P and Q of L with $P \bigcap F = \phi = Q \bigcap F$, $P \subseteq Q \Leftrightarrow \overline{P} \subseteq \overline{Q}.$ Let R be a prime ideal of $\frac{L}{\varphi_F}$. Put $P = \left\{ x \in L \,|\, \frac{x}{\varphi_F} \in R \right\}$. Then P is a prime ideal of L disjoint with F and $\overline{P} = R$. Thus the map $P \mapsto \overline{P}$ is an order isomorphism from the set of prime ideals of L disjoint with F onto the set of prime ideals of $\frac{L}{\varphi_F}$.

Any two ideals I and J are said to be co-maximal, if I $\vee$ J = L. An ADL_0 L is relatively normal [3], if given x, y $\in$ L with x $\leq$ y, the interval [x, y] = {z $\in$ L | x $\leq$ z $\leq$ y} is normal, or equivalently, any two in comparable prime ideals are co-maximal. In general every relatively normal ADL_0 is normal but not conversely. For consider L = {0, a, b, c, d, 1}.



Then the prime ideals P_1 = {0, a, b}, P_2 = {0, a, c} are incomparable and are not co-maximal.

The following is a consequence of the above.

Theorem 2.5: L is normal (relatively normal) if and only if, for any filter F of L, $\frac{L}{\varphi_F}$ is normal (relatively normal).

Note that any dense ADL_0 is normal, because {0} is the unique prime ideal of L. Since the intersection of all minimal prime ideals is {0} in any ADL_0 (see Theorem 1.5), it follows that an ADL_0 L is dense if and only if it has only one minimal prime ideal which is {0}. The following result is another characterization of the normality of an ADL_0.

Theorem 2.6: L is normal if and only if, for any prime filter P of L, $\frac{L}{\varphi_P}$ is a dense ADL_0.

Proof: Suppose that L is normal. Let P be a prime filter of L and I = L – P. Then I is a prime ideal of L and hence I contains a unique minimal prime ideal. Say M. Then M is the only minimal prime ideal of L which is disjoint with P. Therefore, by Theorem 2.4, $\frac{L}{\varphi_P}$ has a unique minimal prime ideal which implies that $\left\{\frac{0}{\varphi_P}\right\}$ is a prime ideal in $\frac{L}{\varphi_P}$ and hence $\frac{L}{\varphi_P}$ is dense. Conversely, suppose that $\frac{L}{\varphi_P}$ is a dense ADL_0, for any prime filter P of L. That is, the zero ideal in $\frac{L}{\varphi_P}$ is prime. Let I be a prime ideal of L. Then P = L – I is a prime filter of L. By our assumption, $\frac{L}{\varphi_P}$ has only one minimal prime ideal namely, $\left\{\frac{0}{\varphi_P}\right\}$. By Theorem 2.4, there is only one minimal prime ideal of L disjoint with P (or, equivalently, contained in I). Thus L is normal.

Given a congruence θ on L, we say that $\frac{L}{\theta}$ is an almost chain, if for any $x, y \in$ L, $\frac{x}{\theta} \wedge \frac{y}{\theta} = \frac{y}{\theta}$ or $\frac{y}{\theta} \wedge \frac{x}{\theta} = \frac{x}{\theta}$.

Theorem 2.7: L is relatively normal if and only if $\frac{L}{\varphi_P}$ is an almost chain, for any prime filter P of L.

Proof: Suppose that L is relatively normal. Let P be a prime filter of L. Suppose that a and b are elements in L such that $\frac{a}{\theta} \wedge \frac{b}{\theta} \neq \frac{b}{\theta}$ and $\frac{b}{\theta} \wedge \frac{a}{\theta} \neq \frac{a}{\theta}$. Then there exists a prime ideal R and S of $\frac{L}{\varphi_P}$ such that $\frac{(a \wedge b)}{\varphi_P} \in R$, $\frac{b}{\varphi_P} \notin R$, $\frac{(b \wedge c)}{\varphi_P} \in S$ and $\frac{a}{\varphi_P} \notin S$.

Put A = $\left\{x \in L | \frac{x}{\varphi_P} \in R\right\}$ and B = $\left\{x \in L | \frac{x}{\varphi_P} \in S\right\}$. Then A and B are prime ideals of L which are disjoint from P and hence A and B are contained in the prime ideal L – P. This implies that A and B are not co-maximal. Also, we have

$$a \wedge b \in A,\ b \notin A \ \&\ a \in A \text{ and } b \wedge a \in B,\ a \notin B \ \&\ b \in B$$

So that A and B are incomparable. This is a contradiction. Thus $\frac{L}{\varphi_P}$ is an almost chain.

Conversely, suppose that, for any prime filter P of L, $\frac{L}{\varphi_P}$ is an almost chain. Let I and J be two incomparable prime ideals of L such that $I \vee J \neq L$. Since every proper ideal is contained in a prime ideal (by the Zorn's lemma), there exists a prime ideal K of L such that $I \vee J \subseteq K$. Put $P = L - K$. Then P is a prime filter of L. Choose $x \in I - J$ and $y \in J - I$. Since $\frac{L}{\varphi_P}$ is an almost chain, with out loss of generality we can suppose that $\frac{x}{\varphi_P} \wedge \frac{y}{\varphi_P} = \frac{y}{\varphi_P}$. Then $(x \wedge y, y) \in \varphi_P$. Therefore $t \wedge x \wedge y = t \wedge y$ for some $t \in P = L - K$. Since $x \in I$, we get that $t \wedge y \in I$. Which is a contradiction, since $t \notin I$ and $y \notin I$. Therefore any two incomparable prime ideal are co-maximal. Hence L is relatively normal.

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