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ZERO-FREE REGIONS FOR POLYNOMIALS WITH RESTRICTED COEFFICIENTS

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ABSTRACT

According to a famous result of Enestrom and Kakeya, if $P(z)=a_n z^n + a_{n-1}z^{n-1}+......+a_1 z+a_0$ *is a polynomial of degree n such that* $0<a_n \le a_{n-1}\le ......\le a_1 \le a_0$, *then P(z) does not vanish in* $|z|<1$. *In this paper we relax the hypothesis of this result in several ways and obtain zero-free regions for polynomials with restricted coefficients and thereby present some interesting generalizations and extensions of the Enestrom-Kakeya Theorem.*

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1. INTRODUCTION AND STATEMENT OF RESULTS

The following elegant result on the distribution of zeros of a polynomial is due to Enestrom and Kakeya [6] :

Theorem A: If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

$a_n \ge a_{n-1}\ge ......\ge a_1 \ge a_0 >0$, then P(z) has all zeros in $|z|\le 1$.

Applying the above result to the polynomial $z^n P(\frac{1}{z})$, we get the following result:

Theorem B: If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

$0<a_n \le a_{n-1}\le ......\le a_1 \le a_0$, then P(z) does not vanish in $|z|<1$.

In the literature [1-5, 7, 8], there exist several extensions and generalizations of the Enestrom-Kakeya Theorem. Recently B. A. Zargar [9] proved the following results:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be **a** polynomial of degree n . If for some real number $k\ge 1$,

$0<a_n \le a_{n-1}\le ......\le a_1 \le ka_0$, then P(z) does not vanish in the disk

$$|z|<\frac{1}{2k-1}.$$

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n . If for some real number $\rho, 0\le \rho < a_n$,

$0<a_n-\rho \le a_{n-1}\le ......\le a_1 \le a_0$, then P(z) does not vanish in the disk

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$$|z| \le \frac{1}{1+\frac{2\rho}{a_0}}.$$

Theorem E: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be **a** polynomial of degree n . If for some real number $k \ge 1$,

$ka_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0$, then P(z) does not vanish in

$$|z| < \frac{a_0}{2ka_n - a_0}.$$

Theorem F: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n . If for some real number $\rho \ge 0,$,

$a_n + \rho \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0$, then P(z) does not vanish in the disk

$$|z| \le \frac{a_0}{2(a_n+\rho)-a_0}.$$

In this paper we give generalizations of the above mentioned results. In fact, we prove the following results:

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n . If for some real numbers $k \ge 1$ and $\rho \ge 0,$,

$a_n - \rho \le a_{n-1} \le ...... \le a_1 \le ka_0$, then P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{k\left(a_0+|a_0|\right)-|a_0|+2\rho-a_n+|a_n|}.$$

Remark 1: Taking $0=\rho<a_n$, Theorem 1 reduces to Theorem C and taking k=1 and $0 \le \rho < a_n,$, it reduces to Theorem D.

Theorem 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be **a** polynomial of degree n . If for some real numbers $\rho \ge 0$ and $0<\tau \le 1$,

$a_n + \rho \ge a_{n-1} \ge ...... \ge a_1 \ge \tau a_0$, then P(z) does not vanish in

$$|z| < \frac{|a_0|}{2\rho+a_n+|a_n|-\tau\left(a_0+|a_0|\right)+|a_0|}.$$

Remark 2: Taking $\tau=1$ and $a_0>0$, Theorem 1 reduces to Theorem F and taking $\tau=1, a_0>0$ and $\rho=(k-1)a_n, k\ge 1$, it reduces to Theorem E.

Also taking $\rho=(k-1)a_n, k\ge 1$, we get the following result which reduces to Theorem E by taking $a_0>0$ and $\tau=1$.

Theorem 3: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n . If for some real numbers $k \ge 1, 0<\tau \le 1$,

$ka_n \ge a_{n-1} \ge ...... \ge a_1 \ge \tau a_0$, then P(z) does not vanish in the disk

$$|z| < \frac{a_0}{2ka_n+(1-2\tau)a_0}.$$

2. PROOFS OF THE THEOREMS

Proof of Theorem 1: We have

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 .$$

Let

$$Q(z) = z^n P\left(\frac{1}{z}\right)$$

and

$$F(z) = (z-1)Q(z) .$$

Then

$$F(z) = (z-1)(a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n)$$
$$= -a_0 z^{n+1} - [(a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + ... + (a_{n-2} - a_{n-1}) z^2 + (a_{n-1} - a_n) z + a_n] .$$

For $|z| > 1$,

$$|F(z)| \geq |a_0||z|^{n+1} - \left[|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + ... + |a_{n-1} - a_n||z| + |a_n| \right]$$
$$= |a_0||z|^n \left[|z| - \frac{1}{|a_0|} \left\{ |a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + ... + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n} \right\} \right]$$
$$> |a_0||z|^n \left[|z| - \frac{1}{|a_0|} \left\{ |ka_0 - a_1 - ka_0 + a_0| + |a_1 - a_2| + ... + |a_{n-1} - a_n + \rho - \rho| + |a_n| \right\} \right]$$
$$\geq |a_0||z|^n \left[|z| - \frac{1}{|a_0|} \left\{ (ka_0 - a_1) + (k-1)|a_0| + (a_1 - a_2) + ... + (a_{n-2} - a_{n-1}) + (a_{n-1} - a_n + \rho) + \rho + |a_n| \right\} \right]$$
$$= |a_0||z|^n \left[|z| - \frac{1}{|a_0|} \left\{ k(a_0 + |a_0|) - |a_0| - a_n + |a_n| + 2\rho \right\} \right]$$
$$> 0$$

if $$|z| > \frac{1}{|a_0|} \left[k\left(a_0 + |a_0|\right) - |a_0| - a_n + |a_n| + 2\rho \right].$$

This shows that all the zeros of F(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|} \left[k\left(a_0 + |a_0|\right) - |a_0| - a_n + |a_n| + 2\rho \right].$$

But those zeros of F(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of F(z) and hence Q(z) lie in

$$|z| \leq \frac{1}{|a_0|} \left[k\left(a_0 + |a_0|\right) - |a_0| - a_n + |a_n| + 2\rho \right].$$

Since $P(z) = z^n Q(\frac{1}{z})$, it follows, by replacing z by $\frac{1}{z}$, that all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{k\left(a_0 + |a_0|\right) - |a_0| - a_n + |a_n| + 2\rho} .$$

Hence P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{k\left(a_0 + |a_0|\right) - |a_0| - a_n + |a_n| + 2\rho} .$$

That proves Theorem 1.

Proof of Theorem 2: We have

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 .$$

Let

$$Q(z) = z^n P(\frac{1}{z})$$

and

$$F(z) = (z-1)Q(z) .$$

Then

$$\begin{aligned} F(z) &= (z-1)\left(a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n\right) \\ &= -a_0 z^{n+1} - \left[(a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + ... + (a_{n-2} - a_{n-1}) z^2 + (a_{n-1} - a_n) z + a_n\right] \end{aligned}$$

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For $|z| > 1$,

$$\begin{aligned} |F(z)| &\geq |a_0||z|^{n+1} - \left[|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + ... + |a_{n-1} - a_n||z| + |a_n|\right] \\ &= |a_0||z|^n \left[|z| - \frac{1}{|a_0|}\left\{|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + ... + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\right\}\right] \\ &> |a_0||z|^n \left[|z| - \frac{1}{|a_0|}\left\{|\tau a_0 - a_1 - \tau a_0 + a_0| + |a_1 - a_2| + ... + |a_{n-1} - a_n + \rho - \rho| + |a_n|\right\}\right] \\ &= |a_0||z|^n \left[|z| - \frac{1}{|a_0|}\left\{(a_1 - \tau a_0) + (1-\tau)|a_0| + (a_2 - a_1) + ... + (a_n + \rho - a_{n-1}) + \rho + |a_n|\right\}\right] \\ &= |a_0||z|^n \left[|z| - \frac{1}{|a_0|}\left\{|a_0| - \tau\left(a_0 + |a_0|\right) + a_n + |a_n| + 2\rho\right\}\right] \\ &> 0 \end{aligned}$$

if

$$|z| > \frac{1}{|a_0|}\left\{|a_0| - \tau\left(a_0 + |a_0|\right) + a_n + |a_n| + 2\rho\right\} .$$

This shows that all the zeros of F(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}\left\{|a_0| - \tau\left(a_0 + |a_0|\right) + a_n + |a_n| + 2\rho\right\} .$$

But those zeros of F(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of F(z) and hence Q(z) lie in

$$|z| \leq \frac{1}{|a_0|}\left\{|a_0| - \tau\left(a_0 + |a_0|\right) + a_n + |a_n| + 2\rho\right\} .$$

Since $P(z) = z^n Q(\frac{1}{z})$, it follows, by replacing z by $\frac{1}{z}$, that all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{|a_0| - \tau\left(a_0 + |a_0|\right) - a_n + |a_n| + 2\rho} .$$

Hence P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_0| - \tau\left(a_0 + |a_0|\right) - a_n + |a_n| + 2\rho} .$$

That proves Theorem 2.

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