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SOME NEW GENERALIZED SUMMATION FORMULAE FOR BASIC HYPERGEOMETRIC FUNCTIONS LEIBNITZ RULE IN q-FRACTIONAL CALCULUS

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ABSTRACTS

In the present paper we try to find some known or new summation formulae for basic hyper geometric functions of one and more variables, using certain fundamental results of q-fractional calculus in the line of Purohit S. D.[5].

1. INTRODUCTION

In the present paper we try to find some known or new summation formulae for basic hyper geometric functions of one and more variables, using certain fundamental results of q-fractional calculus. We have by Agarwal R. P. [1]

$$D_{z,q}^{\beta}(z^{\alpha-1}) = \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)} z^{\alpha-\beta-1}, \ \mathrm{Re}(\alpha) > 0, \text{ for all } \beta. \tag{1.1}$$

q- extension of the Leibnitz rule for the fractional q-derivatives for a product of two functions n terms of a series involving fractional q-derivatives of the individual function as:

$$D_{z,q}^{\beta}\{U(z)V(z)\} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+1)}{2}} (q^{-\beta};q)_n}{(q;q)_n} D_{z,q}^{\beta-n}\{U(zq^n)\} D_{z,q}^{n}\{V(z)\}, \tag{1.2}$$

where $U(z)$ and $V(z)$ are regular functions such that

$U(z) = \sum_{r=0}^{\infty} a_r z^r$, $|z|<R_1$; $V(z) = \sum_{r=0}^{\infty} b_r z^r$, $|z|<R_2$,

Then for result (1.2) $|z|<R = \min(R_1, R_2)$.

We will use the following notations:

For real and complex a, $0<|q|<1$, the q-shifted factorial is defined as:

$$(a;q)_n \equiv (q^a;q)_n = \begin{Bmatrix} 1 & ; if\ n=0 \\ (1-q^a)(1-q^{a+1})\dots(1-q^{a+n-1}) & ; if\ n \in N \end{Bmatrix} \tag{1.3}$$

In terms of q-gamma function (1.3) can be written as:

$$(q^a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \ n > 0 \tag{1.4}$$

where q-gamma function [gasper] is given by

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty (1-q)^{a-1}} \tag{1.5}$$

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Indeed it is easy to verify that

$$\lim_{q\to 1^-} \Gamma_q(a) = \Gamma(a) \text{ and } \lim_{q\to 1^-} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n, \tag{1.6}$$

where $(a)_n = a(a+1)\ldots\ldots\ldots.(a+n-1)$. (1.7)

The Generalized Basic Hypergeometric series ${}_r\Phi_s(\cdot)$, is defined by:

$${}_r\Phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r; \\ b_1, b_2, \ldots, b_r; \end{matrix} q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_n}{(q,b_1,\ldots,b_s;q)_n} z^n \tag{1.8}$$

for convergence 0 < q < 1, for all z if r ≤ s, and |z| < 1 if r = s+1.

Basic Appell function $\Phi^{(1)}(\cdot)$ defined as:

$$\Phi^{(1)}[a, b, b'; c; q, x, y] = \sum_{m,n=0}^{\infty} \frac{(a;q)_{m+n}(b;q)_m(b';q)_n}{(c;q)_{m+n}(q;q)_m(q;q)_n} x^m y^n \tag{1.9}$$

where |x| < 1,|y| < 1

Basic hypergeometric function $\Phi_D^{(3)}(\cdot)$

$$\Phi_D^{(3)}(\cdot)[a, b, b'b''; c; q, x, y, z] = \sum_{m,n,p=0}^{\infty} \frac{(a;q)_{m+n+p}(b;q)_m(b';q)_n(b'';q)_p}{(c;q)_{m+n+p}(q;q)_m(q;q)_n(q;q)_p} x^m y^n z^p \tag{1.10}$$

Basic Lauricell function $\Phi_D^{(n)}(\cdot)$

$$\Phi_D^{(n)}(\cdot)[a,b_1,b_2,..,b_n;c;q,x_1,x_2,..,x_n] = \sum_{m_1,m_2,\ldots,m_n=0}^{\infty} \frac{(a;q)_{m_1+..+m_n}}{(c;q)_{m_1+..+m_n}} \prod_{j=1}^{n} \left\{\frac{(b_j;q)_{m_j} x_j^{m_j}}{(q;q)_{m_j}}\right\} \tag{1.11}$$

And the convergence $|x_1| < 1, |x_2| < 1, \ldots, |x_n| < 1$.

The q- multinomial theorem due to Gasper G. & Rahman M. [4]

$$(a_1, a_2, \ldots, a_{m+1})_n = \sum_{\substack{k_1,k_2,\ldots,k_m \geq 0 \\ n \geq k_1+k_2+\cdots+k_m}} \frac{(q;q)_n a_2^{k_1} a_3^{k_1+k_2} a_{m+1}^{k_1+k_2+\cdots+k_m}}{(q;q)_{k_1}(q;q)_{k_2}\ldots\ldots(q;q)_{k_m}(q;q)_{n-(k_1+k_2+\cdots+k_m)}} \times (a_1;q)_{k_1}(a_2;q)_{k_2}(a_m;q)_{k_m}(a_{m+1};q)_{n-(k_1+k_2+\cdots+k_m)} \tag{1.12}$$

where m = 1,2, … , n = 0,1,2,… .

Purohit S. D.[5] has found some known summation formulae associated with the basic hypergeometric functions and some new summation formulae for the basic Appell function $\Phi^{(1)}(\cdot)$, the basic hypergeometric function $\Phi_D^{(3)}(\cdot)$ and the basic Lauricella function $\Phi_D^{(n)}(\cdot)$ as the application of the q-Leibnitz rule for the fractional order q-derivatives of a product of two basic function .

The results obtained by assigning particular values to the function U (z) and V(z) in the q-Leibnitz rule and result (1.1) as follows:

1) If $U(z) = z^{c-a-1}$ and $V(z) = z^{-(b1+b2+\ldots.+bm)}$ and β = -a

Replacing in (1.2) and using (1.1) result will be Lauricell function $\Phi_D^{(n)}(\cdot)$ i.e.

$$\frac{\Gamma_q(c-a-b_1-b_2-\cdots-b_m)}{\Gamma_q(c-b_1-\cdots-b_m)}\frac{\Gamma_q(c)}{\Gamma_q(c-a)} = \Phi_D^{(m)}(q^a, q^{b_1}, q^{b_2}, \ldots, q^{b_m}; q^c; q, q^{c-a-b_1} q^{c-a-b_1-b_2}, q^{c-a-b_1-b_2-\cdots-b_m}) \tag{1.13}$$

Which is q-analogue due to Gaira M. K. & Dhami H. S. [2] as:

$$\frac{\Gamma(c-a-b_1-b_2-\cdots-b_m)}{\Gamma(c-b_1-\cdots-b_m)}\frac{\Gamma(c)}{\Gamma(c-a)} = F_D^{(m)}(a, b_1, b_2, \ldots, b_m, c; 1) \tag{1.14}$$

2) If $U(z) = z^{c+n-1}$ and $V(z) = z^{-(b1+b2+\ldots.+bm)}$ and β = n, replacing in (1.2) and using (1.1) , we get

$$\frac{(q^{c-b_1-b_2-\ldots-b_m};q)_n}{(q^c;q)_n} = \Phi_D^{(m)}(q^{-n}, q^{b_1}, q^{b_2}, \ldots, q^{b_m}; q^c; q, q^{c+n-b_1} q^{c+n-b_1-b_2}, q^{c+n-b_1-b_2-\cdots-b_m}) \tag{1.15}$$

Which is a q-identity if the parameters are so restricted that each of the functions involved exists.

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2. MAIN RESULTS

In the line of Purohit S.D. [5], we will find the following results

1) If U(z) = $z^{c_1+c_2+..+c_m-a-1}$, V(z) = $z^{-(b1+b2+....+bm)}$ and β = -a

Replacing in (1.2) and using (1.1) ,result will be Lauricell function $\Phi_D^{(n)}(\cdot)$ i.e.

$$\frac{\Gamma_q(c_1+c_2+..+c_m-a-b_1-b_2-\cdots-b_m)}{\Gamma_q(c_1+c_2+..+c_m-b_1-\cdots-b_m)}\frac{\Gamma_q(c_1+c_2+..+c_m)}{\Gamma_q(c_1+c_2+..+c_m-a)}=$$
$$\Phi_D^{(m)}(q^a,q^{b_1},q^{b_2},\ldots,q^{b_m};\ q^{c_1+c_2+..+c_m}\ ;q,q^{c_1+c_2+..+c_m-a-b_1}q^{c_1+c_2+..+c_m-a-b_1-b_2},q^{c_1+c_2+..+c_m-a-b_1-b_2-\cdots-b_m}) \tag{2.1}$$

which is a new result.

2) If U(z) = $z^{c_1+c_2+..+c_m+n-1}$, V(z) = $z^{-(b1+b2+....+bm)}$ and β = n

Replacing in (1.2) and using (1.1) , result will be Lauricell function $\Phi_D^{(n)}(\cdot)$ i.e.

$$\frac{(q^{c_1+c_2+..+c_m-b_1-b_2-..-b_m};q)_n}{(q^c;q)_n} =$$
$$\Phi_D^{(m)}(q^{-n},q^{b_1},q^{b_2},\ldots,q^{b_m};q^{c_1+c_2+..+c_m}\ ;q,q^{c_1+c_2+..+c_m+n-b_1},q^{c_1+..+c_m+n-b_1-b_2},q^{c_1+..+c_m+n-b_1-b_2-\cdots-b_m}) \tag{2.2}$$

which is a new result.

3) If U(z) = $z^{c_1-c_2-..-c_m-a-1}$, V(z) = $z^{-(b1-b2-....-bm)}$ and β = -a

Replacing in (1.2) and using (1.1) result will be

$$\frac{\Gamma_q(c_1-c_2-..-c_m-a-b_1+b_2+\cdots+b_m)}{\Gamma_q(c_1-c_2-..-c_m-b_1+\cdots+b_m)}\frac{\Gamma_q(c_1-c_2-..-c_m)}{\Gamma_q(c_1-c_2-..-c_m-a)}=$$
$$\Phi_D^{(m)}(q^a,q^{b_1},q^{-b_2},\ldots,q^{-b_m};\ q^{c_1-..-c_m}\ ;q,q^{c_1-..-c_m-a-b_1}q^{c_1-..-c_m-a-b_1+b_2},q^{c_1-..-c_m-a-b_1+\ \ldots+b_m}) \tag{2.3}$$

Which is q-analogue to known result due to Gaira M. K. & Dhami H. S. [2]

$$\frac{\Gamma(c_1-c_2-..-c_m-a-b_1-b_2-\cdots-b_m)}{\Gamma(c_1-c_2-..-c_m-b_1-\cdots-b_m)}\frac{\Gamma(c_1-c_2-..-c_m)}{\Gamma(c_1-c_2-..-c_m-a)}=\mathrm{F}_A^{(m)}\ (a,b_1,b_2,\ldots,b_m\ ,c_1-c_2-..-c_m;1) \tag{2.4}$$

Proof of results (2.1), (2.2) and (2.3)

i) If U(z) = $z^{c\ -a-1}$, V(z) = z^{-b} and β = -a

Replacing in (1.2) and using (1.1) we get

$$\frac{\Gamma_q(c-a-b)}{\Gamma_q(c-b)} = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}q^{n(c-a)}(q^a;q)_n}{(q;q)_n}\frac{\Gamma_q(c-a)}{\Gamma_q(c+n)}\frac{\Gamma_q(1-b)}{\Gamma_q(1-b-n)} \tag{2.5}$$

After some algebra (2.5) reduces a summation formula for basic hypergeometric function

$_2\Phi_1(.)$ as : $$\frac{\Gamma_q(c-a-b)}{\Gamma_q(c-a)}\frac{\Gamma_q(c)}{\Gamma_q(c-b)} = {}_2\Phi_1(q^a,q^b;q^c;q,q^{c-a-b}) \tag{2.6}$$

Which is known as q-Gauss summation theorem.

ii) If U(z) = z^{c+n-1} , V(z) = z^{-b} and β = n

Replacing in (1.2) and using (1.1) result will be

$$\frac{(q^{c-b};q)_n}{(q^c;q)_n} = {}_2\Phi_1(q^{-n},q^b;q^c;q,q^{n+c-b}) \tag{2.7}$$

Known as terminating q-Gauss summation formula.

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iii) If U(z) = $z^{\;c_1+c_2-a-1}$, V(z) = $z^{-(b_1+b_2)}$ and β = -a

Replacing in (1.2) and using (1.1) we get

$$\frac{\Gamma_q(c_1+c_2-a-b_1-b_2)}{\Gamma_q(c_1+c_2-b_1-b_2)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}} q^{n(c_1+c_2-a)}(q^a;q)_n}{(q;q)_n} \frac{\Gamma_q(c_1+c_2-a)}{\Gamma_q(c_1+c_2+n)} \frac{\Gamma_q(1-b_1-b_2)}{\Gamma_q(1-b_1-b_2-n)} \tag{2.8}$$

After some algebra (2.8) becomes

$$\frac{\Gamma_q(c_1+c_2-a-b_1-b_2)}{\Gamma_q(c_1+c_2-a)} \frac{\Gamma_q(c_1+c_2)}{\Gamma_q(c_1+c_2-b_1-b_2)} = \sum_{n=0}^{\infty} \frac{q^{n(c_1+c_2-a)}(q^a;q)_n (q^{b_1+b_2};q)_n}{(q;q)_n q^{n(b_1+b_2)}(q^{c_1+c_2};q)_n} \tag{2.9}$$

Using relation (1.12) equation (2.9) becomes

$$\frac{\Gamma_q(c_1+c_2-a-b_1-b_2)}{\Gamma_q(c_1+c_2-a)} \frac{\Gamma_q(c_1+c_2)}{\Gamma_q(c_1+c_2-b_1-b_2)} = \sum_{n\geq k_1} \frac{q^{n(c_1+c_2-a)}(q^a;q)_n}{(q;q)_n q^{n(b_1+b_2)}(q^{c_1+c_2};q)_n} \sum_{k_1\geq 0} \frac{(q;q)_n}{(q;q)_{k_1}(q;q)_{n-k_1}} (q^{b_2})^{k_1}(q^{b_1};q)_{k_1}(q^{b_2};q)_{n-k_1} \tag{2.10}$$

After some algebra (2.10) reduces to basic Appell hypergeometric function (1.9) i.e.

$$\frac{\Gamma_q(c_1+c_2-a-b_1-b_2)}{\Gamma_q(c_1+c_2-a)} \frac{\Gamma_q(c_1+c_2)}{\Gamma_q(c_1+c_2-b_1-b_2)} = \Phi^{(1)}[\,q^a, q^{\,b_1}, q^{\,b_2}; q^{\,c_1+c_2}; q, q^{\,c_1+c_2-a-b_1}, q^{c_1+c_2-a-b_1-b_2}] \tag{2.11}$$

where $0 < |q| < 1$.

iv) If U (z) = $z^{\,c_1+c_2+n-1}$, V (z) = $z^{-(b_1+b_2)}$ and β = n

Replacing in (1.2) and using (1.1) result will be terminating summation formula for basic Appell hypergeometric function

$$\frac{(q^{c_1+c_2-b_1-b_2};q)_n}{(q^{c_1+c_2};q)_n} = \Phi^{(1)}[\,q^{-n}, q^{\,b_1}, q^{\,b_2}; q^{\,c_1+c_2}; q, q^{\,n+c_1+c_2-b_1}, q^{n+c_1+c_2-b_1-b_2}] \tag{2.12}$$

v) Also if U(z) = $z^{\,c_1-c_2-a-1}$, V(z) = $z^{-(b_1-b_2)}$ and β = -a

Replacing in (1.2) and using (1.1) and (1.9) result will be

$$\frac{\Gamma_q(c_1-c_2-a-b_1+b_2)}{\Gamma_q(c_1-c_2-b_1+b_2)} \frac{\Gamma_q(c_1-c_2)}{\Gamma_q(c_1-c_2-a)} = \Phi^{(1)}(q^a, q^{b_1}, q^{-b_2};\; q^{c_1-c_2}; q, q^{c_1-c_2-a-b_1} q^{c_1-c_2-a-b_1+b_2}) \tag{2.13}$$

vi) If U(z) = $z^{\;c_1+c_2+..+c_m-a-1}$, V(z) = $z^{-(b_1+b_2+..+b_m)}$ and β = -a

Replacing in (1.2) and using (1.1) we get

$$\frac{\Gamma_q(c_1+..+c_m-a-b_1-\cdots-b_m)}{\Gamma_q(c_1+..+c_m-b_1-b_2-b_3)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}} q^{n(c_1+\cdots+c_m-a)}(q^a;q)_n}{(q;q)_n} \frac{\Gamma_q(c_1+\cdots+c_m-a)}{\Gamma_q(c_1+\cdots+c_m+n)} \frac{\Gamma_q(1-b_1-\cdots-b_m)}{\Gamma_q(1-b_1-\cdots-b_m-n)} \tag{2.14}$$

After some algebra (2.14) becomes

$$\frac{\Gamma_q(c_1+\cdots+c_m-a-b_1-\cdots-b_m)}{\Gamma_q(c_1+\cdots+c_m-a)} \frac{\Gamma_q(c_1+\cdots+c_m)}{\Gamma_q(c_1+\cdots+c_m-b_1-\cdots-b_m)} = \sum_{n=0}^{\infty} \frac{q^{n(c_1+\cdots+c_m-a)}(q^a;q)_n (q^{b_1+\cdots+b_m};q)_n}{(q;q)_n q^{n(b_1+\cdots+b_m)}(q^{c_1+\cdots+c_m};q)_n} \tag{2.15}$$

Using relation (1.12) equation (2.15) becomes

$$\frac{\Gamma_q(c_1+..+c_m-a-b_1-..-b_m)}{\Gamma_q(c_1+..+c_m-a)} \frac{\Gamma_q(c_1+\cdots+c_m)}{\Gamma_q(c_1+..+c_m-b_1-..-b_m)} = \sum_{n\geq k_1+k_2+\cdots+k_{m-1}} \frac{q^{n(c_1+\cdots+c_m-a)}(q^a;q)_n}{(q;q)_n q^{n(b_1+\cdots+b_m)}(q^{c_1+\cdots+c_m};q)_n} \sum_{k_1+k_2+\cdots+k_{m-1}\geq 0} \frac{(q;q)_n (q^{b_2})^{k_1}(q^{b_3})^{k_1+k_2}\ldots\ldots(q^{b_m})^{k_1+k_2+\cdots+k_{m-1}}}{(q;q)_{k_1}\ldots\ldots(q;q)_{k_{m-1}}(q;q)_{n-(k_1+k_2+\cdots+k_{m-1})}} \times (q^{b_1};q)_{k_1}\ldots(q^{b_{m-1}};q)_{k_{m-1}}(q^{b_m};q)_{n-(k_1+k_2+\cdots+k_{m-1})} \tag{2.16}$$

On further simplification the equation (2.16) becomes a summation formula for the basic Lauricell function (1.11)

$$\frac{\Gamma_q(c_1+..+c_m-a-b_1-..-b_m)}{\Gamma_q(c_1+..+c_m-a)}\frac{\Gamma_q(c_1+\cdots+c_m)}{\Gamma_q(c_1+..+c_m-b_1-..-b_m)} = \Phi_D^{(n)}[q^a, q^{b_1}, \ldots, q^{b_m}; q^{c_1+\cdots+c_m}; q, q^{c_1+..+c_m-a-b_1}, q^{c_1+..+c_3-a-b_1-b_2}, \ldots, q^{c_1+..+c_m-a-b_1-..-b_m}] \quad (2.17)$$

For convergence $0 < |q| < 1$.

vii) If $U(z) = z^{c_1+..+c_m+n-1}$, $V(z) = z^{-(b_1+b_2+b_3)}$ and $\beta = n$

Replacing in (1.2) and using (1.1) result will be

$$\frac{(q^{c_1+..+c_m-b_1-..-b_m};q)_n}{(q^{c_1+..+c_m};q)_n} = \Phi_D^{(n)}[q^{-n}, q^{b_1}, \ldots, q^{b_m}; q^{c_1+..+c_3}; q, q^{n+c_1+..+c_3-b_1}, q^{n+c_1+..+c_3-b_1-b_2}, q^{n+c_1+..+c_3-b_1-..-b_m}] \quad (2.18)$$

This is a new q-identity if the parameters are so restrict that each of the function involved exist.

viii) Also if $U(z) = z^{c_1-\cdots-c_m-a-1}$, $V(z) = z^{-(b_1-\cdots-b_m)}$ and $\beta = -a$

Replacing in (1.2) and using (1.1) result will be

$$\frac{\Gamma_q(c_1-\cdots-c_m-a-b_1+\cdots+b_m)}{\Gamma_q(c_1-\cdots-c_m-b_1+\ldots+b_m)}\frac{\Gamma_q(c_1-\cdots-c_m)}{\Gamma_q(c_1-\cdots-c_m-a)} = \Phi_D^{(n)}(q^a, q^{b_1}, q^{-b_2}, \ldots, q^{-b_m}; q^{c_1-..-c_3}; q, q^{c_1-..-c_3-a-b_1}, q^{c_1-..-c_3-a-b_1+b_2}, q^{c_1-..-c_3-a-b_1+..+b_m}) \quad (2.19)$$

Which is q-analogue due to Gaira M. K. & Dhami H. S. [2].

3. CONCLUSION

The q- Leibnitz rule given by Agrwal R.P. [1] is certainly an important tool for driving many summation formulae which involve various basic hypergeometric functions of one or more variables. The results derived in this paper are general and have certain applications in theory of general hypergeometric function.

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