Research Journal of Pure Algebra -3(1), 2013, Page: 62-66
Available online through www.rjpa.info ISSN 2248–9037

THE INDEPENDENT TRANSVERSAL NEIGHBOURHOOD NUMBER OF A GRAPH

[1]P. M. SHIVASWAMY* & [2]N. D. SONER
[1]*Department of Mathematics, B.M.S College of Engineering Bangalore – 560019, India*
[2]*Department of Studies in Mathematics, University of Mysore, Mysore – 570006, India*

(Received on: 08-12-12; Revised & Accepted on: 23-01-13)

ABSTRACT

A set S of vertices in a graph **G** *is a neighbourhood set of G if G =* $\bigcup_{v\in S}\langle N[v]\rangle$ *where* $\langle N[v]\rangle$ *is the subgraph of G induced by v and all points adjacent to v. A neighbourhood set* $S\subseteq V$ *of a graph G is said to be an independent transversal neighbourhood, if S intersects every maximum independent set of G. The minimum cardinality of an independent transversal neighbourhod set of G is called the independent transversal neighbourhood number of G and is denoted by* $\eta_{it}(G)$*. In this paper we begin an investigation of this parameter.*

Keywords: *Neighbourhood set, Independent set, Independent transversal neighbourhood set.*

2010 Mathematics Subject classification: *05C.*

INTRODUCTION

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to the book by Chartrand and Lesniak [1].

In a graph $G = (V, E)$, the open neighbourhood of a vertex $v\in V$ is $N(v) = \{x \in V : vx \in E\}$, the set of vertices adjacent to v. The closed neighbourhood is $N[v] = N(v) \bigcup \{v\}$. A clique in a graph G is a complete subgraph of G. The maximum order of clique in G is called the clique number and is denoted by $\omega(G)$ and clique of order $\omega(G)$ is called a maximum clique. The subgraph induced by a set $S \subseteq V$ is denoted by $\langle S\rangle$.

A set $D\subseteq V$ is a dominating set, if every vertex in $V – D$ is adjacent to a vertex in D and the minimum cardinality of a dominating set is called the dominating number of G and is denoted by $\gamma(G)$. A survey of advanced topics in domination is given in the book by Haynes *et.al* [3].

Let deg (v) be the degree of vertex v and as usual $\delta(G)$, the minimum degree and $\Delta(G)$, the maximum degree of a graph. $\alpha_0(G)$ is the minimum number of vertices in a vertex cover of G. $\beta_0(G)$ is the minimum number of vertices in a maximal independent set of vertex of G. we employ the notation $\lceil x\rceil$ to denote the smallest integer greater than or equal to x, and $\lfloor x\rfloor$ to denote the largest integer less than or equal to x.

A dominating set $S\subseteq V$ of a graph G is said to be an independent transversal dominating set, if S intersects every maximum independent set of G. The minimum cardinality of an independent tranversal dominating set of G is called the independent transversal domination number of G and is denoted by $\gamma_{it}(G)$. This concept was introduced by Hamid in [2].

A set S of vertices in a graph G is a neighbourhood set of G if $G = \bigcup_{v\in S}\langle N[v]\rangle$ Where $\langle N[v]\rangle$ is a subgraph of G induced by v and all points adjacent to v. The neighbourhood number $\eta(G)$ of a graph G equals the minimum number of vertices in a neighbourhood set of G [4]. In this paper we introduced another basic neighbourhood parameter namely independent transversal neighbourhood number and initiate the study of this new neighbourhood parameter.

A neighbourhood set $S\subseteq V$ of a graph G is said to be an independent transversal neighbourhood, if S intersects every maximum independent set of G. The minimum cardinality of an independent transversal neighbourhood set of G and is called an independent transversal neighbourhood number and denoted by $\eta_{it}(G)$.

***Corresponding author:** [1]*P. M. SHIVASWAMY**
[1]*Department of Mathematics, B.M.S College of Engineering Bangalore – 560019, India*

2. RESULTS
The following results are immediate

Proposition: 1
(i): For any graph G, $\gamma(G) \le \eta(G) \le \eta_{it}(G)$.
(ii): For any graph G, $\gamma(G) \le \gamma_{it}(G) \le \eta_{it}(G)$.

Proposition: A [4] For any path P_p of order p, $\eta(P_p) = \left\lfloor \frac{p}{2} \right\rfloor$

Theorem: 2 For any path P_p of order P, $\eta_{it}(P_p) = \left\lfloor \frac{p}{2} \right\rfloor + 1$.

Proof: Let $P_p = (v_1, v_2 \ldots v_p)$. Then $S = \{V_{2i} : 1 \le i \le 2n\}$ is the n-set of P_p
Further, $\langle V - S \rangle = \left\lfloor \frac{p}{2} \right\rfloor k_1$, and hence every independent set in $V - S$, contains at most $\left\lfloor \frac{p}{2} \right\rfloor$ vertices. Now since $\left\lfloor \frac{p}{2} \right\rfloor = \beta(P_p)$. It follows that $S \cup \{x\}$, where $x \in V–S$ is an independent transversal neighbourhood set of P_p.

$$\text{Hence } \eta_{it}(P_p) = |S| + 1.$$
$$= \frac{p}{2} + 1.$$
$$\text{Since } \eta(\mathrm{P_p}) = \left\lfloor \frac{p}{2} \right\rfloor$$

Proposition B. [4]**:** For any cycle C_p, with $p \ge 4$, $\eta(C_p) = \left\lceil \frac{p}{2} \right\rceil$

Theorem 3: For any cycle C_p of order P, $\eta_{it}(C_p) = \left\lceil \frac{p}{2} \right\rceil + 1$.

Proof: Let $C_p = (v_1, v_2 \ldots v_p)$. Then we consider two cases.

Case (i): If P is odd then $S = \{v_1, v_3, v_5 \ldots v_{p-1}\}$
i.e., $S = \{v_{2i+1}: 0 \le i \le \frac{p-1}{2}\}$ is a n-set of C_p.

Now, since $\langle v - s \rangle = \left\lceil \frac{p}{2} \right\rceil k_1$ every independent set in $V–S$ contains at most $\frac{p}{2}$ vertices and hence $V – S$ contains β_0-set.

Thus it follows that $S \bigcup \{u\}$ where $u \in$ V-S is an independent tranversal neighbourhood set of C_p.

$$\text{Hence } \eta_{it}(C_p) = \eta(C_p) + 1$$
$$= \left\lceil \frac{p}{2} \right\rceil + 1$$

Case (ii): If P is even, then
$s = \left\{ v_{2i} : 1 \le i \le \frac{p}{2} \right\}$ is a n-set of C_p. and $\langle V - S \rangle = \left(\frac{P}{2} \right) k_1$, hence every independence set in $V – S$ contains $\frac{p}{2}$ vertices so that $V – S$ contains β_0-set.

Thus it follows that $S \bigcup \{u\}$ where $u \in V–S$ is an independent transversal neighbourhood set of C_p.

$$\text{hence } \eta_{it}(C_p) = \left|S \bigcup \{u\}\right| = n(C_p) + 1 = \frac{p}{2} + 1.$$

From case (i) and (ii), we have $\eta_{it}(C_p) = \left\lceil \frac{p}{2} \right\rceil + 1.$

Corollary 1: For any wheel W_p on p vertices ($P \neq 4$) $\eta_{it}(W_p) = \left\lceil \frac{p-1}{2} \right\rceil + 1.$

Proof: Clearly, $\eta_{it}(W_p) = \eta_{it}(C_{p-1})$.

$$= \left\lceil \frac{p-1}{2} \right\rceil + 1.$$

Theorem 4: For any graph G, we have $\eta(G) \leq \eta_{it}(G) \leq \eta(G) + \delta(G)$.

Proof: Since an independent transversal neighbourhood set of G is a neighbourhood set, it follows that $\eta(G) \leq \eta_{it}(G)$.

Now let u be a vertex in G with deg $(u) = \delta(G)$, and let S be a n-set in G. Then every maximum independent set of G contains a vertex of $N(u)$, so that $S \bigcup N[u]$ is an independent transversal neighbourhood set of G. Also, since S intersect $N[u]$ it follows that $\left|S \bigcup N(u)\right| \leq \eta(G) + \delta(G)$ and hence the right inequalities follows.

Theorem 5: If G is a disconnected graph, with components $G_1, G_2 \ldots G_r$, then

$$\eta_{it}(G) = \min_{1\leq i\leq r} \left\{ \eta_{it}(G_i) + \sum_{j=1, j\neq i}^{r} \eta(G_j) \right\}.$$

Proof: Let $G = \bigcup_{i=1}^{r} G_i$

Suppose that $N_1, N_2, \ldots, N_r$ are the maximum neighbourhood sets of the graphs $G_1, G_2, \ldots, G_r$ respectively and $S_1, S_2 \ldots S_r$ are the minimum independent transversal neighbourhood sets of $G_1, G_2, \ldots, G_r$.

Let $B_1, B_2 \ldots, B_r$ be the maximum independent sets of $G_1, G_2, \ldots, G_r$. Any independent transversal neighbourhood set S_i, $i = 1, \ldots, r$ is intersect the set $\bigcup_{i=1}^{r} B_i$.

Hence $S_1 \bigcup_{i=2}^{r} N_i$, $S_2 \bigcup_{i=1, i\neq 2}^{r} N_i \ldots S_r \bigcup_{j=1, j\neq r}^{r-1} N_j$ are all the independent transversal neighbourhood of G and the order of those sets will be,

$$\eta_{it}(G_1) + \sum_{i=2\, j\neq 1}^{r} \eta(G_j)$$

$$\eta_{it}(G_2) + \sum_{i=1\, j\neq 2}^{r} \eta(G_j)$$

$$\eta_{it}(G_3) + \sum_{i=1\, j\neq 3}^{r} \eta(G_j) \qquad \vdots \qquad \vdots$$

$$\eta_{it}(G_i) + \sum_{i=1\, j\neq i}^{r} \eta(G_j)$$

Since the minimum independent transversal neighbourhood set is one of the set $S_1\bigcup_{i=1}^{r} N_i$ which has the minimum cardinality.

Hence $\eta_{it}(G) = \min_{1\le i\le r} \left\{\eta_{it}(G_i) + \sum_{j=1,\, j\ne i}^{r} \eta(G_j)\right\}$.

Theorem 6: For any non-complete graph G with clique number ω, $\eta_{it}(G) \le \eta - \omega + 1$. Further equality holds if and only if $\beta_0(G) = 2$.

Proof: Let H be a maximum clique in G. Let $u \in V(H)$. Then $S = V(G) - V(H\text{-}u)$ is neighbourhood set of G. Since $\beta_0(G) \ge 2$ and H is a maximum clique, it follows that every maximum independent set of G intersects S. Hence S is an independent transversal neighbourhood set so that $\eta_{it}(G) \le n - \omega + 1$.

Suppose $\eta_{it}(G) \le \eta - \omega + 1$. Let H be a maximum clique in G. Let u and v be two adjacent vertices such that $u \in V(H)$ and $V \in V(G) - H(G)$. Then $N = \{u\} \bigcup [V(G) - V(H \bigcup v)]$ is a neighbourhood set of G with $|N| = \eta - \omega$. Since $\eta_{it}(G) = n - \omega + 1$, there exists a β_0- set in G such that $N \bigcap S = \phi$ Hence S consists of the vertex u and a vertex $\omega \ne u$ in H, so that $\beta_0(G) = 2$.

Theorem 7: Let l and m be two positive integers with $m \ge 2l - 1$. Then there exist a graph G on m vertices, such that $\eta_{it}(G) = l + 1$.

Proof: Let m = $2l + r$, $r \ge -1$ and let H be any connected graph on l vertices.

Let $V(H) = \{v_1, v_2 \ldots v_l)$. Let G be a graph obtained from H by attaching $r + 1$ pendant edges at V_1 and one edge at each V_i for $i \ge 2$. Let u_i $(i \ge 2)$ be the pendent vertex in G adjacent to V_i clearly $\eta(G) = l$ and $S = \{v_1, v_2, \ldots, v_l, u_i\}$ (u_i is any pendent vertex in G) is a η_{it} set of G. Further every maximal independent set of G intersect S and hence $\eta_{it}(G) = |S| = l + 1$, also $|V(G)| = b$.

Theorem 8: For any non-complete graph G with $\delta(G) \ge 2$, we have $\eta_{it}(G) \le \alpha_0(G)$.

Proof: Let S be a β_0 set of G. Then $V - S$ is a neighbourhood set of G, since, $G \ne K_2$ and $\delta(G) \ge 2$, there exists a vertex v in $V - S$ such that $|N(v) \bigcap S| \ge 2$. Let u and w be two neighbours of V in S, since $\delta(G) \ge 2$, it follows that every neighbourhood of v in S is adjacent to at least one vertex other than v in $V - S$ and hence $D = (V - S) - \{v\}$ is a neighbourhood set of $G - \{v\}$.

Then $D \bigcup \{w\}$ is an independent transversal neighbourhood set of G. This is because $(S - \{w\}) \bigcup \{v\}$ is the only set in the compliment of $D \bigcup \{w\}$, which is not an independent set, and hence $\eta_{it}(G) \le \eta - \beta_0(G) = \alpha_0(G)$.

Corollary 2: If G is a non-complete graph $\eta_{it}(G) = \alpha_0(G) + 1$, then $\eta(G) = \alpha_0(G)$.

Proof: Suppose $\eta_{it}(G) = \alpha_0(G) + 1$. It follows from theorem 8 and 4 that $\eta_{it}(G) \le \eta(G) + 1$ and hence $\alpha_0(G) \le \eta(G)$.

Also since it is always true that $\eta(G) \le \alpha_0(G)$,

we have $\eta(G) = \alpha_0(G)$.

Theorem 9: For any graph G, $1 \le \eta_{it}(G) \le p$. Further $\eta_{it}(G) = p$, if and only if $G = K_p$ or $\overline{K_p}$.

Proof: The inequalities are trivial. Suppose $\eta_{it}(G) = p$. assume $G = K_p$ or $\overline{K_p}$. Then G has at least three vertices u, v and w such that u and v are adjacent and w is not adjacent to one of u, v.

Suppose w is not adjacent to u. This implies that $V - \{u\}$ is an independent transversal n-set of G and hence $\eta_{it}(G) \le p - 1$, which is a contradiction. This proves necessity, sufficient is obvious. The following are some interesting open problem

(1) Characterize graphs for which

(i) $\gamma_{it}(G) = \eta_{it}(G)$

(ii) $\gamma(G) = \gamma_{it}(G) = \eta(G) = \eta_{it}(G)$.

ACKNOWLEDGE
This research was supported by UGC-SAP, DRS 1. No. F. 510/2/DRS/2011 (SAP).

REFERENCES
[1]. G. Chartrand and L. Lesniak, Graphs and Diagrams (Fourth edition. CRC press, Boca Raton, 2005).

[2]. I.S Hamid, Independent transversal Domination in Graphs, Discussions Mathematicae Graph Theory 32(2012) 5-17.

[3]. T. W. Haynes, S.T. Hedeniemi and P.J. Slater, Domination in graphs: Advanced topics (Marcel Dekker, New York, 1998).

[4]. E. Sampath Kumar and P.S Neeralagi (1985). "The neighbourhood number of a graph". Indian J. Pure, Appl. math. 16(2), 126 – 132.

Source of support: UGC-SAP, DRS 1. No. F. 510/2/DRS/2011 (SAP), India, Conflict of interest: None Declared

